General Relativity 2021/22 - Homework 1

Given: 15/10, due: 29/10.

1. In four-dimensional Minkowski spacetime, define new coordinates (t', x', y', z') by

$$t = \left(\frac{c}{g} + \frac{x'}{c}\right) \sinh\left(\frac{gt'}{c}\right) , \quad x = c\left(\frac{c}{g} + \frac{x'}{c}\right) \cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g} , \quad y = y' , \quad z = z' , \tag{1}$$

where g is a constant with the dimensions of acceleration.

   (a) *(1 pt)* Compute the line element of special relativity in terms of these new coordinates.

   (b) *(1 pt)* For $gt'/c << 1$ show that this corresponds to a transformation to a uniformly accelerated frame in Newtonian mechanics.

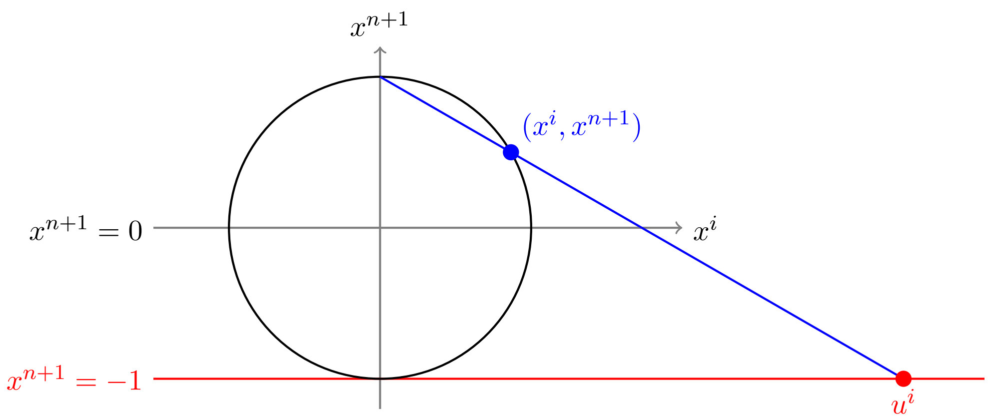
   (c) *(2 pts)* Show that an at-rest clock in this frame at $x' = h$, $h > 0$, runs fast compared to a clock at $x' = 0$ by a factor $(1 + gh/c^2)$. How is this related to the equivalence principle?

2. Consider the n-sphere defined in $\mathbb{R}^{n+1}$ by the equation $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$.

   (a) (*2 pts*) Show that the stereographic projection from the north pole $(x^1, \ldots, x^n, x^{n+1}) = (0, \ldots, 0, 1)$ to the plane $x^{n+1} = -1$ leads to the map

$$\phi_S : (x^1, \ldots, x^n, x^{n+1}) \mapsto u^i \equiv \frac{2x^i}{1 - x^{n+1}} , \tag{2}$$

   defined on $U_S \equiv S^n \backslash \{0, \ldots 0, 1\}$, where u^i are the Cartesian coordinates on the plane. For a given point on the sphere with coordinates $(\vec{x}, x^{n+1})$, where $\vec{x} = (x^1, \ldots, x^n)$, the below diagram shows the geometry in the two-dimensional plane corresponding to the directions $(\hat{n}, x^{n+1})$, where $\hat{n} = \vec{x}/|\vec{x}|$ is the n-dimensional unit vector in the direction picked out by $\vec{x}$.



(b) (*2 pts*) Similarly, the stereographic projection from the south pole $(x^1, \ldots, x^n, x^{n+1}) = (0, \ldots, 0, -1)$ to the plane $x^{n+1} = +1$ leads to the map

$$\phi_N : (x^1, \ldots x^n, x^{n+1}) \mapsto \tilde{u}^i \equiv \frac{2x^i}{1 + x^{n+1}} , \tag{3}$$

defined on $U_N \equiv S^n \backslash \{0, \ldots 0, -1\}$ (no need to show this). Show that on the overlap $U_N \cap U_S$ that the transition function $\phi_N \circ \phi_S^{-1}$ defines a smooth map

$$\tilde{u}^i(u) = \frac{4u^i}{(u^1)^2 + \ldots (u^n)^2} , \tag{4}$$

and hence these give an atlas for the n-sphere.

(c) (*2 pts*) Express the embedding coordinates (x^i, x^{n+1}) of the sphere in $\mathbb{R}^{n+1}$ in terms of the u^i and hence show that the metric on the sphere in the chart U_S is

$$ds^2 = \frac{\delta_{ij} du^i du^j}{(1 + \frac{1}{4}\delta_{kl} u^k u^l)^2} \tag{5}$$