Course 224

Scholarship Questions

1. Prove the Hamilton-Cayley Theorem.

2. Let T be a linear operator on a finite dimensional vector space. Show, without using the Hamilton-Cayley Theorem, that there is a non-zero polynomial f such that $f(T) = 0$. Show that $f(\lambda) = 0$ if λ is an eigenvalue of T.

3. State and prove the Spectral Theorem for a normal operator on a finite dimensional Hilbert Space.

4. State and prove the Primary Decomposition Theorem. Outline the application to the theory of an ordinary homogeneous linear differential equation with constant coefficients.

5. Prove that a system of homogeneous linear equations of rank r on an n-dimensional vector space has an $n-r$-dimensional solution space.

6. Let M be a vector space with a scalar product which is non-degenerate on a finite-dimensional subspace N. Prove that $M = N \oplus N'$, where N' is the orthogonal complement of N.

7. Prove Sylvester's Theorem on the number of plus and minus signs in a quadratic form.

8. Let T be a linear operator on a complex vector space such that $T^3 = 1$. Prove that T is diagonalisable.

9. Let M be a vector space of finite dimension n and basis $u_1 \dots u_n$. Prove that the coordinate functions $u^1 \dots u^n$ form a basis for the dual space M^*. Prove that $u^i \otimes u_j$ is a basis for the tensor product $M^* \otimes M$. Define contraction of tensors and prove that it is well-defined independent of any choice of basis used in the definition.

10. Let M be a finite dimensional vector space with a symmetric scalar product. Show that there is a basis such that the scalar product has a diagonal matrix.

11. Let T be a linear operator on a vector space of finite dimension n such that $T^4 = 0$, but T^3 is non-zero. Find the maximum rank T can have.

12. Let M be a Minkowski space. Show that the transition matrix between Lorentz bases in M is a Lorentz matrix. Show that an isometry of M has a Lorentz matrix with respect to a Lorentz basis. Let u, w be non-zero vectors in M with $(u|u) < 0$ and $(u|w) = 0$. Prove that $(w|w) > 0$.

13. Let $u_1 \dots u_n$ be linearly independent vectors in a vector space M and let $y_1 \dots y_r$ generate M. Prove that $n \leq r$.

14. Let L and N be 4-dimensional vector subspaces of a 7-dimensional vector space. Prove that $L \cap N$ contains a non-zero vector.

15. Let A be a matrix with entries in a field. Prove that the dimension of the row space of A is the same as the dimension of the column space.

16. Let T be a linear operator with Jordan form J. Prove that T can be represented by the transpose of J.

17. Prove the Schwarz inequality in a Hilbert space

18. Let P,Q, be self-adjoint operators in a Hilbert Space, satisfying commutation relation $PQ - QP = \alpha$. Prove the uncertainty relation,
$$(\Delta P)(\Delta Q) \geq \frac{|\alpha|}{2}$$

19. If T is a linear operator on M, and M has an orthonormal basis of eigenvectors of T, prove that T is normal.

20. Let A be a Hermitian matrix. Prove there is a unitary matrix such that PAP^{-1} is diagonal.

21. Let G, A be real symmetric n by n matrices with G positive definite. Prove there is a matrix P such that $P^tGP = I$ and P^tAP is diagonal.

22. Let S, T be commuting diagonalisable operators. Prove that they are simultaneously diagonalisable.

23. Let T be a diagonalisable operator on a finite dimensional vector space M. Let N be a vector subspace of M which is invariant under T. Prove that T_N is diagonalisable.

24. If a linear operator T has matrix A, prove that rank $T =$ rank A.

25. Let T be an invertible linear operator on a vector space M of finite dimension n. Prove that powers of T, $(T^r | r \in Z)$ generate a subspace of Hom(M) of dimension at most n.

26. Let T be a linear operator on a finite dimensional vector space M. Show that there is a unique monic polynomial q of minimal degree such that $q(T) = 0$. Show that T is diagonalisable if and only if q is a product of distinct linear factors.

27. If $T : L \to M$ and $S : M \to N$ are linear operators, prove that rank $ST \geq \text{rank}S + \text{rank}T - \text{dim}M$. If $T : M \to M$ is a linear operator, and rank $T =$ rank T^2, prove that the intersection of ker T and Im T contains only the zero vector.

28. If $T : M \to N$ is a linear operator, prove that dim ker T + dim Im T = dim M

29. Show that the vectors $x_1 \ldots x_r$ are linearly independent if and only if $x_1 \wedge \ldots \wedge x_r$ is non-zero.

30. Show that the linearly independent vectors $x_1 \ldots x_r$ generate the same subspace as the vectors $y_1 \ldots y_r$ if and only if $x_1 \wedge \ldots \wedge x_r$ is a scalar multiple of $y_1 \wedge \ldots \wedge y_r$

31. Show that the matrix equation $Ax = b$ is equivalent to $x_1 c_1 + \ldots + x_n c_n = b$ where the c_i are the columns of A. Use the wedge product to derive Cramer's rule.

32. Show that there is a linear isomorphism $x \mapsto \widetilde{x}$ from M to the dualspace of M^* such that

$$\langle \widetilde{x}, f \rangle = \langle f, x \rangle$$

for all $x \in M$ and $f \in M^*$.

33. State briefly what a category is and give an example.

34. Prove that a nilpotent operator can be represented by a matrix whose entries are all zero or one.

35. Prove that the skew-symmetriser satisfies:

$$A\Big[(AS) \otimes T\Big] = A\Big[S \otimes T\Big]$$

$$A(S \otimes T) = (-1)^{st} A(T \otimes S)$$

36. Let M be a finite dimensional real oriented vector space with non-degenerate symmetric scalar product. Prove that the volume form

$$\text{vol} = u^1 \wedge \ldots \wedge u^n$$

is independent of the choice of standard basis $u_1, \ldots, u_n$.

37. Find a matrix P such that PAP^{-1} is a Jordan matrix, where

$$A = \begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$$

38. Find a matrix P such that PAP^{-1} is a Jordan matrix, where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -5 & 21 & 17 \\ 6 & -26 & -21 \end{pmatrix}$$

39. Find a matrix P such that PAP^{-1} is a Jordan matrix, where

$$A = \begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}$$

40. Find a matrix P such that PAP^{-1} is a Jordan matrix, where

$$A = \begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}$$

41. Find a matrix P such that PAP^{-1} is a Jordan matrix, where

$$A = \begin{pmatrix} 3 & 4 & 3 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

42. Find a matrix P such that PAP^{-1} is a Jordan matrix, where

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 7 & 1 & 2 & 1 \\ -17 & -6 & -1 & 0 \end{pmatrix}$$

43. Show that the push-forward T_* preserves tensor products, commutes with permutations and preserves the wedge product

44. Show how to use the push-forward to establish $\det ST = \det S \det T$

45. Let f be the function defined on the space of non-singular $n \times n$ real matrices and given by

$$f(A) = A^{-1}$$

Prove that f is differentiable and find its derivative.

46. Let f be the function defined on the space of non-singular $n \times n$ real matrices and given by

$$f(A) = A^{-2}$$

Prove that f is differentiable and find its derivative.

47. Let f be a C^2 function of two independent variables. Prove that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

48. Show that the function

$$f(x, y) = 2xy\frac{x^2 - y^2}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is not C^2 on any open set containing $(0, 0)$.

49. Define the Hodge star operator and prove that

$$*u^1 \wedge \ldots \wedge u^r = s_{r+1} \ldots s_n u^{r+1} \wedge \ldots \wedge u^n$$

where $s_i = (u_i|u_i)$.

50. Let F be a real valued homogeneous differentiable function of n real variables. Prove that F is an eigenfunction of the operator

$$\sum_{j=1}^{n} x^j \frac{\partial}{\partial x^j}$$

51. Prove the chain rule for functions on finite dimensional real or complex vector spaces.
52. State and prove the Implicit Function Theorem
53. Prove the pull-back commutes with differentials.
54. Prove the chain rule for maps of manifolds.
55. Define:

    C^∞ manifold

    C^∞ compatible coordinates

    tangent space at a point on a manifold

    velocity vector of a parametrised path

    differential of a function

    partial derivative $\frac{\partial f}{\partial y^i}$ with respect to a coordinate system y^i

    pull-back of a differential form under a map

    push-forward of a tangent vector under a map

    n-dimensional coordinate system