MA2C03 - DISCRETE MATHEMATICS - TUTORIAL NOTES
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Summary: During this tutorial we went through some of the properties and proofs in graph theory presented during the last week of last term.

1 Isomorphisms

If two graphs are 'isomorphic' then they are essentially the same graph. We can formalise this notion with the following definition:

Definition 1.1. An *isomorphism* between two graphs (V, E) and (V', E') is a bijective function $\varphi : V \to V'$ (1) satisfying(2)

$$\forall a, b \in V \text{ with } a \neq b, \text{ the edge } ab \in E \Leftrightarrow \text{ the edge } \varphi(a)\varphi(b) \in E'$$

Implicitly we need three conditions for an isomorphism of graphs:

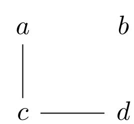
(1) A bijection between the vertices V and V' (also known as a 1-1 correspondence)

(2) A bijection between the edges E and E' (also known as a 1-1 correspondence)

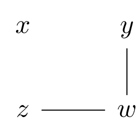
We also need the explicit preservation of edge relation:

$$\forall a, b \in V \text{ with } a \neq b, \text{ the edge } ab \in E \Leftrightarrow \text{ the edge } \varphi(a)\varphi(b) \in E'$$

Example 1.1. Consider the two graphs (V, E):



and (V', E'):



Define φ as follows:

$$\varphi(a) = y \quad \varphi(b) = x \quad \varphi(c) = w \quad \varphi(d) = z$$

This will be an isomorphism; φ is a bijection of vertices and the edges relation is preserved. Note that (V', E') is just (V, E) flipped through a vertical axis, so we would hope that these two graphs are isomorphic as they are essentially the same.

2 Regular graphs

Recall the definition of *vertex degree*:

Definition 2.1. Let (V, E) be a graph. The degree $\deg v$ of a vertex $v \in V$ is defined as the number of edges of the graph that are *incident* to v, i.e. the number of edge with v as one of their endpoints.

This leads to the definition of regular:

Definition 2.2. A graph (V, E) is called *regular* if $\exists k \in \mathbb{N}$ such that every vertex of the graph has degree k.

There are two nice proofs to go with the idea of vertex degrees:

Theorem 2.1. *Let (V, E) be a graph. Then $\sum_{v \in V} \deg v = 2(\#E)$.*

Sketch Proof.
Every edge in E contributes two to the total vertex degree count. ■

Theorem 2.2. *A complete bipartite graph $k_{p,q}$ is regular $\Leftrightarrow p = q$.*

Proof.
$(\Rightarrow)$. Assume $k_{p,q}$ is regular. Then

$$\forall v_1 \in V_1,\ \forall v_2 \in V_2 \quad \deg v_1 = \deg v_2$$

As the graph is complete, every vertex in V_1 is connected to every vertex in V_2, namely $\deg v_1 = \#V_2 = q$ and $\deg v_2 = \#V_1 = p$. Therefore $p = q$ as required.
$(\Leftarrow)$. Assume $p = q$. As the graph is complete, for all $v_1 \in V_1$ and $v_2 \in V_2$,

$$\deg v_1 = \#V_2 = q = p = \#V_1 = \deg v_2$$

making $k_{p,q}$ regular, as required. ■

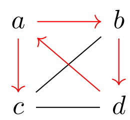
3 Walks, trails and paths

When memorising the terminology, keep this in mind:

- A TRAIL TRAVERSES each EDGE once.
- A PATH PASSES THROUGH each POINT (aka vertex) once.

It's easy to think of examples of walks that are trails but not paths. Consider the following walk on k_4:

$$abdac$$



This is clearly not a path, but is a trail. What about the other way around?

Theorem 3.1. *All paths are trails.*

Proof.
We will prove this by contradiction. Assume there is a path $p = v_{i_0}v_{i_1}\dots v_{i_n}$ which is not a trail. Then it must have a repeated edge ab:

$$p = v_{i_0}v_{i_1}\dots ab\dots ab\dots v_{i_n}$$

or

$$p = v_{i_0}v_{i_1}\dots ab\dots ba\dots v_{i_n}$$

Suppose the repeated edges fall on vertices v_w, v_x, v_y, v_z respectively. There are two cases:

(1) $x = y$, e.g. something like v_0abav_2. In this case $v_w = v_z$ or $v_y = v_z$, so a vertex is repeated in the path.

(2) $x \neq y$, e.g something like v_0abv_2ab. Again, at least one vertex is repeated in the path.

It boils down to the fact that no matter what, a vertex is repeated; a contradiction to the fact p is a path. We conclude the assumption is false; all paths are trails. ■

4 Connected graphs

Definition 4.1. An undirected graph (V, E) is called *connected* if $\forall u, v \in V$, $\exists$ a path in the graph from u to v.

If we wish to prove all graphs are a disjoint union of their connected components, there are 4 steps in the proof:

(1) We can define a relation $\sim$ on vertices by

$$a \sim b \Leftrightarrow \exists \text{ a walk in the graph from } a \text{ to } b.$$

(2) As it turns out, this is an equivalence relation on vertices. Thus the equivalence classes partition the vertex set $V = v_1 \cup \cdots \cup v_n$.

(3) We get for free a partition of the set of edges:

$$e_i = \{ab \in E : a, b \in v_i\}.$$

(4) The subgraphs (v_i, e_i) are connected and disjoint from each other. These are known as the *components* of the graph (V, E).

Of course, items (1)-(4) need to be proven at each step! However what is shown is that a graph can be written as a disjoint union of connected subgraphs. This is a handy property; if you don't understand a graph (V, E), you might have better luck understanding its connected components. If you can do that, then you understand the full graph.

Graphs are currently a hot topic in maths; Prof. Lázsló Babai from the University of Chicago claims to have an algorithm for solving the Graph Isomorphism Problem (essentially, determine if two graphs are isomorphic) in quasipolynomial time. There was an error found in his algorithm, but less than two weeks ago he claims to have found a work around. More information can be found here including photos of his talk at the University of Chicago - unfortunately they didn't take any of me waiting outside, hoping for a seat.