MA2C03 - DISCRETE MATHEMATICS - TUTORIAL NOTES
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Summary: During this tutorial we finished looking through the solutions of Assignment 2 and went through some of the properties and proofs in graph theory presented during the first few weeks of this term.

1 Circuits

Using closed walks we can deduce a surprising number of features of graphs. We begin with a definition:

Definition 1.1. Let (V, E) be a graph. A *circuit* is a (nontrivial) **closed trail** in (V, E). That is, a closed walk with **no repeated edges** (passing through at least two vertices).

Remark 1.1. Use the following mnemonic:
CIRCUIT = CLOSED TRAIL.

Circuits are *simple* if the vertices of the walk (except for the initial and final vertices) are distinct. In this sense it is almost like a path.

The next natural question to ask is *when does a graph have simple circuits? Is there anyway to tell, apart from trial and error?* The lecture notes contain two criteria for simple circuits, which are easily seen once an example graph is drawn:

(1) Every vertex has degree ≥ 2.

(2) $\exists u, v \in V$ such that there exists 2 distinct *paths* from u to v.

NB: Drawing an example graph does NOT constitute a proof. If I handed you a glass of seawater, is that proof the oceans are empty of fish? No!

We'll now talk about a specific kind of trail/circuit: the *Eulerian trail/circuit.*

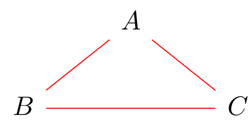
Definition 1.2. An *Eulerian trail* in a graph is a trail that traverses every edge of the graph.
Note: as it is a *trail* it can only traverse edges once. Thus an Eulerian trail is a walk traversing every edge *exactly* once.

Definition 1.3. An *Eulerian circuit* in a graph is a circuit that traverses every edge of the graph.

The main difference between *Definition 1.2* and *Definition 1.3* is an Eulerian circuit is closed; you return to your starting vertex.

Remark 1.2. Use the following mnemonic:
EULERIAN TRAIL = EVERY EDGE is TRAVERSED.
EULERIAN CIRCUITS ARE CLOSED.

Example 1.1. Consider the graph K_3, the connected graph on three vertices:

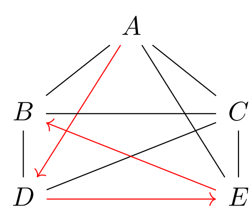


The walk $ABCA$ is an Eulerian circuit; it traverses every edge of the graph once and the start and endpoints coincide.

The main theorem of this section is as follows:

Theorem 1.1. *Let (V, E) be a graph and $v_0 \dots v_n$ be a trail. If $v \in V$ is a vertex, then the number of edges of the trail incident to v is **even**, except when the trail is not closed and the trail starts or finishes wih v, in which case the number of edges incident to v is **odd**.*

This somewhat wordy theorem is pretty simple to understand once we are looking at a graph. Consider K_5, with the trail $ADEB$:



If $v = D$, the number of edges incident (AD, DE) is even and when $v = A$ (say), the number of edges incident is odd (AD). Note if $v = C$ the number of edges incident is 0, which is technically even.

Two important corollaries from this:

Corollary 1.2. *Let v be a vertex of the graph. Given any **circuit** in the graph, the number of edges incident to v traversed by that circuit is even.*

Corollary 1.3. *If a graph admits an **Eulerian circuit**, then the degree of every vertex of that graph must be even.*

Remark 1.3. Note that with the contrapositive of *Corollary 1.3*, the infamous `Seven Bridges of Königsberg` problem is solved! There exists a vertex of odd degree, so the graph doesn't admit an Eulerian circuit.

Now the question becomes; *is the converse true?* As the online notes lay out, we can work through the proof systematically in a series of lemmas:

Lemma 1.1. *If the degree of each vertex is even, then there exists a circuit in the graph.*

Moral of the proof:
Given an edge vw, if we can't find a trail beginning with v and ending at v, then $\deg v$ is odd, a contradiction. ■

We'll tackle the rest next week.