An Analysis of Tame Topology using O-Minimality

Brian Tyrrell
[redacted]

Supervisor: Professor Andreea Nicoara

School of Mathematics
Trinity College Dublin
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Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Abstract

We present various properties of o-minimal structures from a model-theoretic standpoint and demonstrate their merit as a framework for Grothendieck's *tame topology* by comparing o-minimal geometry with semialgebraic and semi/subanalytic geometry. We also demonstrate the power of o-minimality and quantifier elimination by giving model-theoretic proofs of the Łojasiewicz inequality, the Tarski-Seidenberg theorem, Hilbert's 17th Problem, and Hilbert's Nullstellensatz that are obtained easier than their classical counterparts.

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"My approach toward possible foundations for a tame topology has been an axiomatic one. Rather than declaring (which would indeed be a perfectly sensible thing to do) that the desired "tame spaces" are no other than (say) . . . semianalytic spaces, and then developing in this context the toolbox of constructions and notions which are familiar from topology, supplemented with those which had not been developed up to now, for that very reason, I preferred to work on extracting which exactly, among the geometrical properties of the semianalytic sets in a space $\mathbb{R}^n$, make it possible to use these as local "models" for a notion of "tame space" (here semianalytic), and what (hopefully!) makes this notion flexible enough to use it effectively as the fundamental notion for a "tame topology" which would express with ease the topological intuition of shapes. Thus, once this necessary foundational work has been completed, there will appear not one "tame theory", but a vast infinity . . ."

Alexander Grothendieck, *Esquisse d'un Programme* [16].

"Mon approche vers des fondements possibles d'une topologie modérée a été une approche axiomatique. Plutôt que de déclarer (chose qui serait parfaitement raisonnable certes) que les "espaces modérés" cherchés ne sont autres (disons) que les espaces semianalytiques . . . , et de développer dès lors dans ce contexte l'arsenal des constructions et notions familières en topologie, plus celles certes qui jusqu'à présent n'avaient pu être développées et pour cause, j'ai préféré m'attacher à dégager ce qui, parmi les propriétés géométriques de la notion d'ensemble semianalytique dans un espace $\mathbb{R}^n$, permet d'utiliser ceux-ci comme "modèles" locaux d'une notion "d'espace modéré" (en l'occurrence, semianalytique), et ce qui (on l'espère!) rend cette notion d'espace modéré suffisamment souple pour pouvoir bel et bien servir de notion de base pour une "topologie modérée" propre à exprimer avec aisance l'intuition topologique des formes. Ainsi, une fois le travail de fondements qui s'impose accompli, il apparaîtra non une "théorie modérée", mais une vaste infinité . . ."

Contents

1. Introduction

O-minimality (short for *order-minimality*) originally arose in the 1980's through the work of van den Dries [5] and Knight, Pillay and Steinhorn [13, 23] in *model theory*. Model theory is a branch of mathematical logic concerned with studying mathematical structures by examining what is true (from a logical perspective) in these structures, and what subsets of these structures can be defined by first order logical formulae. The latter relates directly to o-minimality, which imposes the condition that the sets defined by logical formulae of one variable in an *ordered* structure must be a finite union of points and intervals. By an *ordered* structure we mean a structure with some order $<$ imposed on its elements. As we shall see, o-minimality is a very nice condition for a structure to meet and the properties of o-minimal structures are well organised, simple, and practical. The reason we can make the claim of *practicality* is connected to our underlying field of study in logic. Chang and Keisler [4] in their 1990 book begin in the introduction with the equation

$$\text{universal algebra} \ + \ \text{logic} \ = \text{model theory}$$

in an effort to capture the relation of model theory to both abstract logic and abstract algebra, two branches of mathematics that seem to have their own, disconnected, goals. Model theory is thus immensely practical as through logic and logical formulae we can prove (in many cases, in easier ways than the classical solution) results about algebraic structures used in 'day-to-day' mathematics such as groups, rings, or algebraically closed fields for example. An instance of this practicality is given in §3.2 where we show that when analysing the o-minimal *model* of certain fields we are in fact generalising the study of real semialgebraic geometry. While Knight et al. [13, 23] lay the foundations of o-minimality at a high level of generality for model-theoretic structures, it was van den Dries' work [5] on the field of real numbers with exponentiation (see §2.3) that provided both a framework to follow and a direction in mathematics to explore. This brings us to the second possible viewpoint concerning o-minimality.

In his book *Tame Topology and O-minimal Structures* [6], van den Dries pioneered the use of o-minimality as a tool to analyse the geometry of certain classes of sets. The perspective he took was to define an o-minimal structure (which we refer to as a *o-minimal VDD structure* to prevent a notation clash) from a set theoretic standpoint as follows:

Definition 1.1.

(1) A *VDD structure* on a nonempty set R is a sequence $\mathfrak{S} = (S_m)_{m\in\mathbb{N}}$ such that for each $m \geq 0$:

  (a) S_m is a Boolean algebra of subsets of R^m.

  (b) If $A \in S_m$ then $R \times A$, $A \times R \in S_{m+1}$.

  (c) $\{(x_1, \ldots, x_m) \in R^m : x_1 = x_m\} \in S_m$.

(d) If $A \in S_{m+1}$ then $\pi(A) \in S_m$ where $\pi : R^{m+1} \to R^m$ is the projection map to the first m coordinates.

(2) Let $(R, <)$ be a dense linearly ordered set without endpoints[1]. A VDD structure $\mathfrak{S}$ on $(R, <)$ is called *o-minimal* if it is a VDD structure on R satisfying the additional conditions:

(a) The sets in S_1 are exactly the finite unions of intervals and points.

(b) $\{(x, y) \in R^2 : x < y\} \in S_2$ (that is, the relation $<$ belongs to S_2).

The advantage of this definition is that no model theory is then needed to develop the theory of o-minimal structures from a topological and geometric standpoint, which is helpful as analytic geometers and topologists generally do not have a background in logic. We will in fact prove there is nothing lost in this definition, namely that an o-minimal VDD structure is equivalent to a model-theoretic o-minimal structure (*Theorem 4.14*).

From this viewpoint, o-minimal structures were seen as a way to generalise *semialgebraic* and *subanalytic* geometry. The definitions of a semialgebraic and subanalytic set can be stated as follows:

Definition 1.2. A subset of $\mathbb{R}^n$ is *semialgebraic* if it is a Boolean combination of solution sets of polynomial equations $p(x_1, \ldots, x_n) = 0$ and polynomial inequalities $p(x_1, \ldots, x_n) > 0$.

Definition 1.3. A subset $X \subset \mathcal{M}$ of a real analytic manifold is ***semi**analytic* if and only if for all $a \in \mathcal{M}$ there exists a neighbourhood U of a such that $X \cap U$ is a finite Boolean combination of solution sets of equations $p(x_1, \ldots, x_n) = 0$ and inequalities $p(x_1, \ldots, x_n) > 0$, where p is a real analytic function.

Definition 1.4. A subset $X \subset \mathcal{M}$ of a real analytic manifold is ***sub**analytic* if and only if for all $a \in \mathcal{M}$ there exists a neighbourhood U of a such that $X \cap U$ is the projection of a relatively compact semianalytic set.
That is, there exists a real analytic manifold $\mathcal{N}$ and a relatively compact semianalytic subset Y of $\mathcal{M} \times \mathcal{N}$ such that

$$X \cap U = \pi(Y)$$

where $\pi : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ is the standard projection.

Where o-minimal geometry and semialgebraic/subanalytic geometry come together, however, is in the realm of tame topology. In 1984 Grothendieck first introduced the concept of *tame topology* or *topologie modérée* [16] as a new branch of topology dedicated to the study of the nice or 'tame' properties of semialgebraic and subanalytic sets, properties that are further elaborated on in §8, Bierstone and Milman [2], and

[1] $<$ is a linear order in which if $x < y$ there exists z such that $x < z < y$ (*dense*), and there does not exist y, z such that $y < x$ or $x < z$ for all $x \in R$ (*without endpoints*).

from an o-minimal standpoint in van den Dries and Miller [8], van den Dries [6], Machpherson [17], Kaiser [11] and Marker [19].

We will follow the first interpretation of o-minimal structures, as objects of model theory in our exploration of tame topology, which will allow us to (when necessary) prove powerful results using logic, generalise and apply the theory of o-minimal structures more effectively, and visualise aspects of the theory more readily.

The thesis will develop along the following route:

- In §2 we will present to the reader the relevant model-theoretic definitions and facts for use in the additional sections.
- In §3 we discuss *quantifier elimination* in detail, a tool (related to o-minimality) that allows us to simplify the study of algebraically closed fields and $\mathbb{R}$ and prove the Tarski-Seidenberg theorem, Hilbert's 17th problem, and Hilbert's Nullstellensatz with ease.
- In §4 we formally define an o-minimal structure and prove the equivalence of model-theoretic o-minimal structures and o-minimal VDD structures before giving examples and considering o-minimal ordered groups and rings.
- In §5 we present the Monotonicity and Cell Decomposition theorems, which prove subsets of any dimension defined in o-minimal structures are of a particularly simple format. We also discuss definable families and the Trivialization theorem, which proves the definable maps of certain o-minimal structures can be broken into essentially 'trivial' parts.
- In §6 we prove o-minimal structures admit certain types of *curves* and *paths.* They will play the same part as sequences in a metric space and also will allow us to demonstrate some of the topological properties of o-minimal structures.
- In §7 we give the classical proof of the Łojasiewicz inequality [2] followed by a generalisation obtained in a much more efficient and smooth way with o-minimality.
- In §8 we gather together all the material on o-minimality we have covered to argue that the study of o-minimal geometry is the realisation of tame topology as Grothendieck described it.

2. Model theory

In this section, we wish to present to the reader the mathematical tools we will use to deal with subjects covered in the thesis. Enderton [9] and Marker [20] serve as excellent introductory texts, while Chang & Keisler [4] and Hodges [10] are more advanced.

The main focus of this thesis is to investigate the classes of sets leading to tame topological behaviour by way of model theory, so it is important we develop fully and formally the notions we will use.

2.1. Languages and Structures

Definition 2.1. A *language* L is a collection of symbols divided into three separate groups: *constant* symbols, *relation* symbols, and *function* symbols.

Note that all languages are assumed to contain equality $(=)$ as a relation. Equality is not usually written if listing out the elements of L.

Example 2.2. Common languages that are used in §3 and §7.3 are:

- The *language of rings* $L_r = \{0, 1, +, -, \times\}$, where 0, 1 are constants, and $+, -, \times$ are binary function symbols.
- The *language of ordered rings* $L_{or} = \{0, 1, +, -, \times, <\}$, where $<$ is a binary relation symbol.
- The *language of ordered fields with exponentiation* $L_{\exp} = \{<, 0, 1, +, -, \times, \exp\}$ where $\exp$ is the unary function $\exp : x \mapsto e^x$.

◊

Now that we have the concept of language we need concise interpretations of the languages symbols. To do this we need *somewhere* to interpret the symbols. Together this forms the motivation for defining a *structure*:

Definition 2.3. Let L be a language. An *L-structure* $\mathfrak{A}$ is the data of:

- An underlying set, denoted $|\mathfrak{A}|$, known as the *domain of* $\mathfrak{A}$.
- Interpretations for all symbols in L, meaning:
  - Each constant symbol $c \in L$ is assigned to an element $c^{\mathfrak{A}}$ of $|\mathfrak{A}|$.
  - Each relation symbol $R \in L$ of arity $k < \omega$ is interpreted to hold on some subset $R \subseteq |\mathfrak{A}|^k$, meaning
  $$R^{\mathfrak{A}}(x_1, \dots, x_k) \text{ is true } \Leftrightarrow (x_1, \dots, x_k) \in R.$$
  - Each function symbol $f \in L$ of arity $k < \omega$ is interpreted to take every element of $|\mathfrak{A}|^k$ to an element of $|\mathfrak{A}|$. That is, f is a function
  $$f^{\mathfrak{A}} : |\mathfrak{A}|^k \to |\mathfrak{A}|.$$

The interpretation of any symbol $c, R, f \in L$ in $\mathfrak{A}$ is represented (as we see above) by $c^{\mathfrak{A}}, R^{\mathfrak{A}}, f^{\mathfrak{A}}$.

The best way to illustrate this definition is with a few examples.

Example 2.4. Let $L = \{\leq\}$ be a language containing $\leq$, a binary relation symbol. We wish to make $\mathfrak{A}$ a structure where $|\mathfrak{A}| = \{0, 1\}$. In order to do this, we need to form an interpretation of $\leq$ on $|\mathfrak{A}|$. Define

$$a \leq b \Leftrightarrow (a, b) \in \{(0, 0), (0, 1), (1, 1)\}$$

This interpretation is known as the *usual* interpretation of $\leq$; the interpretation we expect to have for the symbol $\leq$. With the interpretation defined, $\mathfrak{A}$ is a structure (though not very useful or interesting). $\lozenge$

Example 2.5. Let $L = \{0, <, S\}$ be a language, where 0 is a constant, $<$ is a binary relation with the usual interpretation and S a function known as the *successor* function. We wish to make $\mathfrak{B}$ a structure, where $|\mathfrak{B}| = \mathbb{N}$.
Let 0 have the usual interpretation on $\mathbb{N}$ and define S, a unary function, by:

$$S(x) \in \{y : \neg(\exists z(x < z < y))\}$$

Note that $\{y : \neg(\exists z(x < z < y))\}$ will have one element if $<$ is the usual interpretation, so S is well defined. With these interpretations, $\mathfrak{B}$ is a structure representing properties of the natural numbers in the model-theoretic world. $\lozenge$

Example 2.6. Let $L = \{\sim\}$ be a language where $\sim$ is a binary relation symbol. If we wish to study graphs, we can define an L-structure $\mathfrak{N}$ whose domain is $\mathbb{N}^2$, where

$$a \sim b \quad \Leftrightarrow \quad \text{there is an edge between } a \text{ and } b.$$

$\lozenge$

If we want more detailed structures we need a systematic and formal way of specifying what properties a structure does and does not satisfy. This is done through *sentential logic* (also known as *propositional calculus*).

Definition 2.7. Let L be a language containing constant, relation and function symbols.
Terms are one of the following:

- A variable is a term.
- A constant symbol is a term.
- If F is a k-ary function symbol of L and $t_1, \ldots, t_k$ are terms, then $F(t_1, \ldots, t_k)$ is a term.

- A string of symbols is a term if and only if it can be shown to be one by finitely many applications of the above.

Atomic formulae are one of the following:

- "$t_1 = t_2$" is an atomic formula, where t_1 and t_2 are terms.
- If R is a k-ary relation symbol of L and $t_1, \dots, t_k$ are terms, then $R(t_1, \dots, t_k)$ is an atomic formula.

We wish to expand this logic to a predicate logic that can quantify over variables so we can make statements such as "there exists ..." or "for all ...". We will therefore use *first order logic*, which has these properties and uses *formulae* to do so:

Definition 2.8. The *formulae* of L are defined as follows:

- An atomic formula is a formula.
- If ϕ, φ are formulae, so are $\phi \wedge \varphi$, $\phi \vee \varphi$, and $\neg\varphi$.
- If v is a variable and φ a formula, so is $\forall v(\varphi)$.
- A string of symbols is a formula if and only if it may be shown to be so by finitely many applications of the above.

Definition 2.9. We say a variable v is *free* in a formula ϕ if it is not inside a quantifier, such as $\forall v$ or $\exists v$. A *sentence* is a formula with no free variables.

We are ready to begin looking at examples.

Example 2.10. Let X be a set totally ordered by $\leq$, that is:

(1) For all a, b in X, either $a \leq b$ or $b \leq a$.

(2) If $a \leq b$ and $b \leq a$ then $a = b$.

(3) If $a \leq b$ and $b \leq c$ then $a \leq c$.

These statements can be rewritten in the style of first order logic:

(1) $\varphi_1(a, b) = a \leq b \vee b \leq a$

(2) $\varphi_2(a, b) = (a \leq b \wedge b \leq a) \to (a = b)$

(3) $\varphi_3(a, b, c) = (a \leq b \wedge b \leq c) \to (a \leq c)$

The formulae $\varphi_1(a, b)$, $\varphi_2(a, b)$, $\varphi_3(a, b, c)$ can be combined into the encompassing sentence

$$\phi = (\forall a)(\forall b)(\forall c)(\varphi_1(a, b) \wedge \varphi_2(a, b) \wedge \varphi_3(a, b, c))$$

which captures (1) – (3). This is more commonly written as

$$\phi = \forall a, b, c \ \ (\varphi_1(a, b) \wedge \varphi_2(a, b) \wedge \varphi_3(a, b, c))\,.$$

$\diamondsuit$

Remark 2.11. We can also define formulae *with parameters*. For example in the language $L = \{<\}$:

$$\varphi(x, a, b, c) = (a < x < b) \vee (x = c).$$

φ is a formula in one variable, x, and has parameters a, b, c. In general, suppose $\mathfrak{A}$ is an L-structure. Stylistically we write a formula with parameters as $\phi(x, u)$ where $u \in |\mathfrak{A}|^m$ for some m and $\phi(x, u)$ takes values in $|\mathfrak{A}|^n$ for some n. $\Diamond$

Example 2.12. For a more abstract example to demonstrate the broadness of *Definitions 2.7 & 2.8*, consider $L = \{<, +, 0\}$. Note that $+$ is considered a binary function. The following are terms:

(a) 0.

(b) $x + 0$.

We would represent (a) by the formula $t_a(x)$ where $t_a(x)$ is 0 for any input x. Similarly we would represent (b) by $t_b(x) = x + 0$ whose value on x is $x + 0$.
The following are atomic formulae:

- $x = 0$.
- $0 < 0 + x$.

The following are formulae:

- $x = 0$.
- $(0 < 0 + x) \vee \neg(0 < x)$
- $\forall x(x < 0 \wedge 0 < x)$

Note that apart from *Definitions 2.7 & 2.8* there are no restrictions in place to prevent 'nonsense' such as $\forall x(x < 0 \wedge 0 < x)$. Indeed our intuitive idea of what constitutes 'nonsense' needs to be formalised; we need to be able to say which formulae are true and which are false. $\Diamond$

Remark 2.13. Notation: We shall use the following shorthand when dealing with L-structures: $\mathfrak{A} = (|\mathfrak{A}|, \{c_i\}_{i\in I}, \{R_j\}_{j\in J}, \{f_k\}_{k\in K})$ is a structure with domain $|\mathfrak{A}|$ in a language $L = \{\{c_i\}_{i\in I}, \{R_j\}_{j\in J}, \{f_k\}_{k\in K}\}$ where c_i are the constant symbols, R_j are the relation symbols and f_k are the function symbols. $\Diamond$

2.2. Structures and Truth

Now that we know how to express ourselves in this formal language, we need to determine the validity of our statements; after all, there's not much point in having a system of logic that cannot tell true from false! As is often the case, truth (or *satisfaction*) is meaningless without context and structures provide the necessary context.

Definition 2.14. Truth in a structure.

- Suppose t_1 and t_2 are terms. If $\varphi =$ "$t_1 = t_2$" then $a_1, \ldots, a_n$ *satisfies* φ in $\mathfrak{A}$ if $t_1^{\mathfrak{A}}(a_1, \ldots, a_n) = t_2^{\mathfrak{A}}(a_1, \ldots, a_n)$ in $|\mathfrak{A}|$.
  More specifically:
  - If a term t is the constant symbol c, then $t^{\mathfrak{A}}(a_1, \ldots, a_n)$ is $c^{\mathfrak{A}}$, the interpretation of c in $|\mathfrak{A}|$.
  - If t is the variable x_i, then $t^{\mathfrak{A}}(a_1, \ldots, a_n) = a_i^{\mathfrak{A}}$.
  - If t is the k-ary function $F(t_1, \ldots, t_k)$, then

$$t^{\mathfrak{A}}(a_1, \ldots, a_n) = F^{\mathfrak{A}}(t_1^{\mathfrak{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathfrak{A}}(a_1, \ldots, a_n)).$$

- If $\varphi = R(t_1, \ldots, t_k)$, then $a_1, \ldots, a_n$ satisfies φ in $\mathfrak{A}$ if and only if

$$(t_1^{\mathfrak{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathfrak{A}}(a_1, \ldots, a_n)) \in R^{\mathfrak{A}}.$$

The notation "$\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$" is used to represent "$a_1, \ldots, a_n$ satisfies φ in $\mathfrak{A}$". This notation is equivalent to the phrase "$\varphi(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$" or "$\mathfrak{A}$ *models* $\varphi(a_1, \ldots, a_n)$". The satisfaction of L-formulae in $\mathfrak{A}$ is defined inductively as follows:

- Suppose θ_1, θ_2 are formulae of L. If $\varphi = \theta_1 \wedge \theta_2$, then

$$\mathfrak{A} \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \mathfrak{A} \models \theta_1(a_1, \ldots, a_n) \text{ and } \mathfrak{A} \models \theta_2(a_1, \ldots, a_n)$$

- Suppose ϕ is a formula of L. If $\varphi = \neg\phi$ then

$$\mathfrak{A} \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \text{ NOT } \mathfrak{A} \models \phi(a_1, \ldots, a_n)$$

  NOT $\mathfrak{A} \models \phi(a_1, \ldots, a_n)$ is written $\mathfrak{A} \not\models \phi(a_1, \ldots, a_n)$ and is taken to mean "$\phi(a_1, \ldots, a_n)$ is false in $\mathfrak{A}$".

- If $\varphi = (\forall x_i)\phi$ where $\phi = \phi(x_1, \ldots, x_n)$ is an L-formula, then

$$\mathfrak{A} \models \varphi(a_1, \ldots, a_n) \qquad \Leftrightarrow$$
$$\text{for every element } b \in |\mathfrak{A}|, \qquad \mathfrak{A} \models \phi(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$$

Definition 2.15. If $\mathfrak{A}$ is an L-structure, *the theory of* $\mathfrak{A}$:

$$\text{Th}(\mathfrak{A}) = \{\varphi : \varphi \text{ is a sentence of } L \text{ and } \mathfrak{A} \models \varphi\}$$

Two L-structures $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent*, written $\mathfrak{A} \equiv \mathfrak{B}$, if $\text{Th}(\mathfrak{A}) = \text{Th}(\mathfrak{B})$.

Remark 2.16. The phrases *model* and *structure* are often used in conjunction, however we will emphasise the distinction between the two: a *structure* is the mathematical object given by *Definition 2.3*, whereas the term *model* usual refers to a structure and *theory* concurrently (e.g. "$\mathfrak{G}$ is a *model of* the theory of groups"). $\Diamond$

Now that we have defined a structure, the mathematical object with which we have chosen to work, and we have a clear definition of what is and is not true in this structure, the next natural question to ask is *how can two structures be related?*

The answer naturally depends on two things: the *structure* itself (its language, cardinality, etc.) and what is *true* in the structure.

Definition 2.17. Suppose $\mathfrak{A}$, $\mathfrak{B}$ are L-structures. An *L-embedding* $f : \mathfrak{A} \to \mathfrak{B}$ is an injective map $f : |\mathfrak{A}| \to |\mathfrak{B}|$ preserving the interpretation of L, that is:

- If c is a constant of L, then $c^{\mathfrak{A}} \mapsto f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.
- If R is a k-ary relation symbol of L, then
$$(a_1, \dots, a_k) \in R^{\mathfrak{A}} \quad \Leftrightarrow \quad (f(a_1), \dots, f(a_k)) \in R^{\mathfrak{B}}.$$
- If F is a k-ary function symbol,
$$F^{\mathfrak{A}}(a_1, \dots, a_k) = b \quad \Leftrightarrow \quad F^{\mathfrak{B}}(f(a_1), \dots, f(a_k)) = f(b).$$

Definition 2.18. Two L-structures $\mathfrak{A}$, $\mathfrak{B}$ are *isomorphic* (written $\mathfrak{A} \cong \mathfrak{B}$) if there is a bijective L-embedding between them.

In algebra two structures are isomorphic if a bijection between them preserves some property. Here, a bijection between model-theoretic structures preserves truth.

There is a relation between *elementary equivalence* and *isomorphisms*:

Theorem 2.19. *If $\mathfrak{A}$, $\mathfrak{B}$ are L-structures, $\mathfrak{A} \cong \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$.*

Proof.
This theorem might not be so surprising as isomorphisms between L-structures naturally preserve truth. For the converse, however it is not necessarily true that $\mathfrak{A} \equiv \mathfrak{B} \Rightarrow \mathfrak{A} \cong \mathfrak{B}$. Many examples can be constructed of elementarily equivalent structures of different sizes. For example, Enderton [9, Chapter 2] explains that $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ (with the usual interpretations) are elementarily equivalent, and these are clearly not isomorphic. ■

Finally, we should specify how structures can sit inside each other. In group theory, we have subgroups of groups. In linear algebra, we have subspaces of vector spaces. In model theory, we have *substructures* of structures:

Definition 2.20. Let $\mathfrak{A}$, $\mathfrak{B}$ be two L-structures. If $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and the inclusion map is an L-embedding, then $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ and $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$. We usually write $\mathfrak{A} \subseteq \mathfrak{B}$ to denote this relationship.

Example 2.21. Let $L = \{\sim\}$, where $\sim$ is a binary relation on edges defined in *Example 2.6*. Let $\mathfrak{C}$ be an L-structure, where $|\mathfrak{C}| = \{1, 2, 3, 4\}$. We are considering the graphs on vertices $1, 2, 3, 4$.

We can reframe any graph on four vertices as a structure by requiring sentences describing how the graph is connected be true in $\mathfrak{C}$. For example, if we wish $\mathfrak{C}$ to be complete we enforce

$$\mathfrak{C} \models \forall x, y\ (x \neq y \to x \sim y)$$

Furthermore, we can obtain K_3 as a substructure by defining $\mathfrak{D}$ such that $|\mathfrak{D}| = \{1, 2, 3\}$ and the three nodes are maximally connected. The relation $\sim$ on $|\mathfrak{C}|$ restricts to $|\mathfrak{D}|$. The canonical embedding i of $\mathfrak{D}$ into $\mathfrak{C}$ is then an L-embedding, as required (see *fig. 1*). $\diamondsuit$

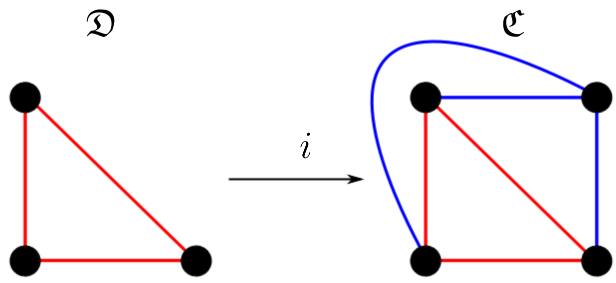


Figure 1

Definition 2.22. $\mathfrak{N}$ is an *elementary* substructure of $\mathfrak{M}$ if $\mathfrak{N} \subseteq \mathfrak{M}$ are structures of the same language L such that for all L-formulae $\varphi(x_1, \ldots, x_n)$ and all elements $a \in |\mathfrak{N}|^n$,

$$\mathfrak{N} \models \varphi(a) \quad \Leftrightarrow \quad \mathfrak{M} \models \varphi(a).$$

Remark 2.23. "$\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$" is written $\mathfrak{N} \prec \mathfrak{M}$. It should be noted that an *elementary* substructure is **not** the same as an *elementarily equivalent* substructure. This follows from an examination of *Definition 2.22*. $\diamondsuit$

There is a test in place to determine if a substructure is elementary, which is known as the *Tarski-Vaught test*:

Lemma 2.24. The Tarski-Vaught test. *Let $\mathfrak{M}$ be an L-structure and $\mathfrak{N}$ a substructure of $\mathfrak{M}$. Then $\mathfrak{N} \prec \mathfrak{M}$ if and only if for all single variable L-formulae $\varphi(x, y_1, \ldots, y_n)$ and all elements $b \in |\mathfrak{N}|^n$, if $\mathfrak{M} \models \exists x\ \varphi(x, b)$ then there is an element $a \in |\mathfrak{N}|$ witnessing this, i.e. $\mathfrak{M} \models \varphi(a, b)$.*

Proof.
See Marker [20, Proposition 2.3.5]. ■

This test is important as it gives a practical test to preform with formulae of one variable, rather than attempting to construct a map to prove one structure can be elementarily embedded into another structure.

2.3. Deduction and Quantifier Elimination

Recall Th($\mathfrak{A}$) ("the theory *of* $\mathfrak{A}$") is the set of sentences true in an L-structure $\mathfrak{A}$. In general we define a *theory* to be a set of sentences. Two small definitions that will prove useful for our understanding of theories and then quantifier elimination are *deduction* and *consistency.*

Definition 2.25. A set of sentences Γ *deduces* a sentence φ, written $\Gamma \vdash \varphi$, if there exists a finite sequence $(\phi_0, \ldots, \phi_n)$ of sentences such that $\phi_n = \varphi$ and each ϕ_i for $0 \leq i \leq n$ is either a member of Γ, a tautology[2], or obtained from two previous sentences by *modus ponens* (from α and $\alpha \to \beta$ infer β).

Example 2.26. We can prove for an L-sentence φ, $\{\varphi\} \vdash \varphi \wedge \varphi$.

Note that it is a tautology that $\varphi \to \varphi \wedge \varphi$ (this can be proven via a truth table). Since we have φ (as an element of $\{\varphi\}$) and $\varphi \to \varphi \wedge \varphi$, by modus ponens we can conclude $\varphi \wedge \varphi$. This is the sentence we wished to obtain, thus the deduction is complete and it is true that $\{\varphi\} \vdash \varphi \wedge \varphi$, as required.

Formally,

(1) $\phi_0 = \varphi$, an element of $\{\varphi\}$.

(2) $\phi_1 = \varphi \to \varphi \wedge \varphi$, a tautology.

(3) By modus ponens on (1), (2), we deduce $\varphi \wedge \varphi$, thus $\phi_3 = \varphi \wedge \varphi$.

$\diamondsuit$

As with the definition of truth in structures we need to ensure we are not writing nonsense when we do not mean to:

Definition 2.27.

- A set of sentences Γ is *inconsistent* if $\Gamma \vdash \varphi \wedge \neg\varphi$ for some sentence φ (i.e. Γ deduces a contradiction).
- A set of sentences is *consistent* if it is not inconsistent.

Finally, there are two theorems that connect consistency and structures:

Theorem 2.28.

(1) **Soundness theorem.** *If Γ is a set of L-sentences and if there is an L-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$, then Γ is consistent.*

(2) **Completeness theorem.** *If a set of L-sentences Γ is consistent, then there exists an L-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$.*

[2] A universally true statement.

Proof.
See Enderton [9, §2.5]. ■

In *Definitions 2.8 and 2.14* we discussed what quantifiers are and how to incorporate them into our understanding of structures. Now we can review a large portion of model theory: how to get rid of quantifiers. *Quantifier elimination* is helpful as it forces the formulae of a language to be of a particularly simple form, a fact we can exploit when discussing the theory of a structure.

First, we will modify our definition of a *theory*:

Definition 2.29. A *theory* T is a **consistent** set of L-sentences.

Example 2.30. The theory RCF is defined as the collection of axioms for a field (in first order logic) and the following sentences:

(1) For each $n \geq 1$, $\forall x_1, \ldots, x_n\ (x_1^2 + \cdots + x_n^2 + 1 \neq 0)$.

(2) $\forall x\ \exists y\ ((y^2 = x) \vee (y^2 + x = 0))$.

(3) For each $n \geq 0$, $\forall x_0, \ldots, x_{2n}\ \exists y\ (y^{2n+1} + \sum_{i=0}^{2n} x_i y^i = 0)$.

♢

Definition 2.31. Let T be a theory in a language L. For an L-formula φ, we say

$$T \models \varphi \qquad \Leftrightarrow \qquad \text{for all models } \mathfrak{A} \text{ of } T,\ \mathfrak{A} \models \varphi.$$

Definition 2.32. Let T be a theory in a language L. Two L-formulae φ, ψ are T-*equivalent* if

$$T \models \forall x_1, \ldots, x_n\ (\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)),$$

which is to say in every model where T is true, φ is true when ψ is true, and φ is false when ψ is false.

Definition 2.33. A theory T admits *elimination of quantifiers* if and only if for every formula φ there is a *quantifier-free formula* ψ such that φ and ψ are T-equivalent.

Example 2.34. Marker [20, §3.1] gives the following familiar example. In the real numbers, a solution to the quadratic formula is an example of quantifier elimination:

$$\mathbb{R} \models (\exists x(ax^2 + bx + c = 0)) \leftrightarrow ((a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0))).$$

Therefore, $\varphi = \exists x(ax^2 + bx + c = 0)$ is $\mathbb{R}$-equivalent to the quantifier-free formula

$$\psi = (a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0)).$$

Similarly in the complex numbers,

$$\mathbb{C} \models (\exists x(ax^2 + bx + c = 0)) \leftrightarrow (a \neq 0 \vee b \neq 0 \vee c = 0)$$

(In fact, both the theories of real closed fields (such as $\mathbb{R}$) or algebraically closed fields (such as $\mathbb{C}$) admit quantifier elimination, so every formula can be replaced with a quantifier-free one in the respective theories. We shall prove this result in §3.1.) ♢

Remark 2.35. If an L-structure admits quantifier elimination, what quantifier-free formula is equivalent to $\exists x(x = x)$? To ensure we always have an answer, we require that the language L must contain a least one constant c so that $\exists x(x = x)$ is equivalent to $c = c$. $\Diamond$

Quantifier elimination will be discussed in more detail in the next section where we examine its proof in different structures and its applications.

Remark 2.36. O-minimality was first introduced by van den Dries [5] when he attacked the problem of quantifier elimination in $(\mathbb{R}, +, \times, \exp)$, where exp is the real exponential function

$$\exp : \mathbb{R} \to \mathbb{R} \qquad x \mapsto e^x.$$

This structure is discussed further in §4.1. $\Diamond$

3. Quantifier Elimination

In this section we will approach geometry from a more logical standpoint and implement more abstract model theory. We will excurse through [20] and prove the Tarski-Seidenberg theorem, Hilbert's 17th Problem and Hilbert's Nullstellensatz by way of logic.

If a structure admits Quantifier Elimination (QE) we can use this property to reap powerful benefits. Often we do not realise that either we are already using QE or that QE can be used to prove an even deeper result. For example, recall the following proposition from linear algebra:

Proposition 3.1. *An $n \times n$ matrix has an inverse if and only if its determinant is nonzero.*

As Buzzard [3] remarks, "the existence of an inverse is n^2 existence statements satisfying n^2 equations, [whereas] the determinant is just one assertion about something being non-zero". This is an example of quantifier elimination in $\mathbb{R}$.

We will concern ourselves with proving two theories admit QE: ACF (Algebraically Closed Fields) and RCF (Real Closed Fields).

To understand what these theories represent we will define what is understood to be an *algebraically closed field* or a *real closed field* from an algebraic standpoint. We begin with terminology regarding ordered algebraic structures:

Definition 3.2.

- An *ordered* group is a group equipped with a linear order that is invariant under left and right multiplication, namely $x < y \Rightarrow zx < zy$ and $xz < yz$.
- An ordered ring is a ring equipped with a linear order such that $0 < 1$, $<$ is translation invariant and $<$ is invariant under multiplication by positive elements.
- An ordered field is an ordered division ring with commutative multiplication.

Definition 3.3. Let F be a field.

- F is *orderable* if there is a linear order $<$ on F making $(F, <)$ an ordered field.
- F is *formally real* if -1 is not a sum of squares.
- F is *real closed* if it is formally real with no proper formally real extensions.

Orderable fields are automatically formally real. For our purposes, we will only state the following three theorems and refer to them as needed:

Theorem 3.4. [20, Theorem 3.3.3] *Formally real fields are orderable; moreover, if $a \in F$ and $-a$ is not a sum of squares of elements of F, then there is an ordering of F where a is positive.* ■

Theorem 3.5. [20, Theorem 3.3.5] *If F is a formally real field, then F is real closed if and only if $\forall a \in F$, a or $-a$ is a square and every polynomial of odd degree has a root.* ■

Theorem 3.6. [20, Lemma B.10] *F is a real closed field if and only if there is an ordering on F such that the Intermediate Value theorem holds for all polynomials over F with nonnegative degree.* ■

Definition 3.7. Recall from *Example 2.30* RCF is defined as the collection of axioms for a field (in first order logic) and the following sentences:

(1) For each $n \geq 1$, $\forall x_1, \ldots, x_n \; (x_1^2 + \cdots + x_n^2 + 1 \neq 0)$.

(2) $\forall x \; \exists y \; ((y^2 = x) \vee (y^2 + x = 0))$.

(3) For each $n \geq 0$, $\forall x_0, \ldots, x_{2n} \; \exists y \; (y^{2n+1} + \sum_{i=0}^{2n} x_i y^i = 0)$.

If $\mathfrak{F} \models$ RCF then by (1) $\mathfrak{F}$ is formally real and by (2), (3) $\mathfrak{F}$ is real closed (*Theorem 3.5*). Therefore if a structure $\mathfrak{F}$ is a model of these sentences, it is a real closed field.

Note that the idea of real closed fields is designed such that $\mathbb{R}$ is real closed.
We continue on to *algebraically closed fields*:

Definition 3.8. A field F is called *algebraically closed* if every non constant polynomial $p(x) \in F[x]$ has a root in F.

Definition 3.9. ACF is defined as the collection of axioms for a field (in first order logic), and for $n = 1, 2, \ldots$ the sentences

$$\forall a_0, \ldots, a_n \; \left(a_n \neq 0 \rightarrow \exists x \; (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0)\right).$$

3.1. Proving Quantifier Elimination

RCF (as a theory in the language $L_{or} = \{0, 1, +, -, \times, <\}$) admits QE. We will prove this statement by parsing the proof ACF (as a theory in the language $L_r = \{0, 1, +, -, \times\}$) admits QE, and then by outlining steps on how to generalise this proof to include the order.

Definition 3.10. Let L be a language and $\mathfrak{A}$ an L-structure.

(1) If $\phi(x_1, \ldots, x_n)$ is an L-formula we say $\mathfrak{A} \models \phi$ if for all $a \in |\mathfrak{A}|^n$ we have $\mathfrak{A} \models \phi(a)$.

(2) If T is a collection of L-formulae and $\mathfrak{A}$ an L-structure, we say $\mathfrak{A} \models T$ if for all $\phi \in T$, $\mathfrak{A} \models \phi$.

Recall *Definition 2.31: $T \models \phi$* if for all L-structures $\mathfrak{A}$ with $\mathfrak{A} \models T$, then $\mathfrak{A} \models \phi$.

We come to the definition we will use for the rest of the section:

Definition 3.11. Let T be an L-theory. Let $\vec{x}$ denote the n-tuple $(x_1, \ldots, x_n)$. The set of *universal consequences* of T,

$$T_\forall := \{\forall \vec{x}\, \varphi(\vec{x}) : \ \varphi(\vec{x}) \text{ is a quantifier-free } L\text{-formula and } T \models \varphi\}.$$

This definition is somewhat inaccessible, but from [3, Lemma 23] we obtain:

Lemma 3.12. *If $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{N} \models T$, then $\mathfrak{M} \models T_\forall$.* ■

This lemma is perhaps not so surprising given that substructures preserve the truth of quantifier-free formulae. A full proof is given in Marker [20, Prop. 1.1.8].

Lemma 3.12 allows us to quickly determine the set of universal consequences for a theory T by passing to a substructure of a model of T. In the case of ACF, the set of universal consequences is the theory of integral domains:

Lemma 3.13. $\mathfrak{I} \models \mathrm{ACF}_\forall$ *if and only if* $\mathfrak{I}$ *is an integral domain.*

Proof.
Let $\mathfrak{I}$ be an integral domain. $\mathfrak{I}$ is a subring of the algebraic closure of its field of fractions $\mathfrak{K}$. By definition $\mathfrak{K} \models \mathrm{ACF}$, thus $\mathfrak{I} \models \mathrm{ACF}_\forall$ by *Lemma 3.12.* On the other hand, suppose $\mathfrak{I} \models \mathrm{ACF}_\forall$. Note our language is $L_r = \{0, 1, +, -, \times\}$, thus $0, 1$ are already named elements in the structure with the properties that

$$\forall x\ (0 + x = x + 0 = x) \qquad \text{and} \qquad \forall x\ (1 \times x = x \times 1 = x).$$

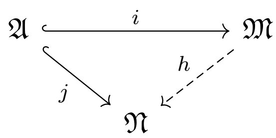
These two sentences are both universal consequences of ACF as are the axioms of distributivity, associativity and commutativity (for both addition and multiplication). The existence of additive inverses follows from the $-$ operation. Thus $\mathfrak{I}$ is a ring. Also, $\mathfrak{I}$ models the sentences

$$0 \neq 1, \qquad \forall x, y\ (xy = yx), \qquad \forall x, y\ (xy = 0 \to x = 0 \vee y = 0)$$

as these are universal consequences of ACF. We conclude $\mathfrak{I}$ is an integral domain, as required. ■

We will need the following definition to tie universal consequences to quantifier elimination:

Definition 3.14. A theory T has *algebraically prime models* if for any $\mathfrak{A} \models T_\forall$, there exists $\mathfrak{M} \models T$ and an embedding $i : \mathfrak{A} \hookrightarrow \mathfrak{M}$ such that for all models $\mathfrak{N} \models T$ and corresponding embeddings $j : \mathfrak{A} \hookrightarrow \mathfrak{N}$ there exists an embedding $h : \mathfrak{M} \to \mathfrak{N}$ such that the following diagram commutes:



We say $\mathfrak{M}$ is an *algebraically prime extension* of $\mathfrak{A}$.

Put simply, given $\mathfrak{A} \models T_\forall$, there exists a model $\mathfrak{M}$ of T such that for all other models $\mathfrak{N}$ of T, where $\mathfrak{A}$ can be embedded into $\mathfrak{N}$, $\mathfrak{M}$ can be embedded into $\mathfrak{N}$.

Definition 3.15. If $\mathfrak{M}, \mathfrak{N}$ are L-models of T with $\mathfrak{M} \subseteq \mathfrak{N}$, we call $\mathfrak{M}$ *simply closed in* $\mathfrak{N}$ (written $\mathfrak{M} \prec_s \mathfrak{N}$) if for all quantifier-free formulae $\varphi(x, u)$ and for all $a \in |\mathfrak{M}|^n$,

$$\mathfrak{N} \models \exists x(\varphi(x,a)) \quad \Rightarrow \quad \mathfrak{M} \models \exists x(\varphi(x,a)).$$

We say T is a *simply closed theory* if for all $\mathfrak{M}, \mathfrak{N}$ models of T with $\mathfrak{M} \subseteq \mathfrak{N}$, $\mathfrak{M}$ is simply closed in $\mathfrak{N}$.

Remark 3.16. Note $\mathfrak{M} \prec \mathfrak{N} \Rightarrow \mathfrak{M} \prec_s \mathfrak{N}$ (and that *Definition 3.15* is reminiscent of the Tarski-Vaught test, *Lemma 2.24*). ♢

These definitions culminate to the following theorem:

Theorem 3.17. *If T is a simply closed theory with algebraically prime models, then T admits QE.*

Proof.
See Marker [20, Corollary 3.1.12]. ■

Definitions 3.11-3.15 are somewhat heavy and technical but in practice are usually proven using algebraic facts, as we shall see in what follows:

Lemma 3.18. ACF *is a simply closed theory.*

Proof.
Let $\mathfrak{M}, \mathfrak{N}$ be algebraically closed fields with $\mathfrak{M} \subseteq \mathfrak{N}$. If $\phi(x, u)$ is a quantifier-free L_r-formula suppose for $b \in |\mathfrak{M}|^k$, $\mathfrak{N} \models \phi(a, b)$ for some $a \in \mathfrak{N}$.

$\phi(x, b)$ is equivalent to a conjunction of atomic formulae and the negations of atomic formulae, which in the language of rings are polynomials. Thus $\phi(x, b)$ is equivalent to:

$$\bigwedge_{i=1}^{n} p_i(x) = 0 \wedge \bigwedge_{j=1}^{m} q_j(x) \neq 0 \qquad p_i, q_j \in \mathfrak{M}[x] \text{ for } 1 \leq i \leq n, 1 \leq j \leq m. \tag{1}$$

We wish to show $\mathfrak{M} \models \exists x \phi(x, b)$, that is, (1) has a solution in $\mathfrak{M}$.

We know there is a solution $a \in \mathfrak{N}$ to

$$\bigwedge_{i=1}^{n} p_i(x) = 0$$

thus (as $\mathfrak{M}$ is algebraically closed) $a \in \mathfrak{M}$. Therefore $\phi(x, b)$ is equivalent to

$$\bigwedge_{j=1}^{m} q_j(x) \neq 0$$

by the fact $\mathfrak{M}$ is algebraically closed. However, each q_j has only finitely many solutions so there are only finitely many 'bad' elements β_l such that $\bigwedge_{j=1}^{m} q_j(\beta_l) = 0$. Algebraically closed fields are necessarily infinite so we can stay away from these 'bad' elements. Therefore there exists some $c \in \mathfrak{M}$ such that $\mathfrak{M} \models \phi(c, b)$. Thus ACF is a simply closed theory as required. ■

Lemma 3.19. ACF *has algebraically prime models.*

Proof.
Suppose $\mathfrak{A} \models \mathrm{ACF}_\forall$, i.e. $\mathfrak{A}$ is an integral domain. If $\mathfrak{M}$ is the algebraic closure of the fraction field of $\mathfrak{A}$, then $\mathfrak{M} \models \mathrm{ACF}$ and $\mathfrak{M}$ embeds into any algebraically closed field $\mathfrak{N}$ containing $\mathfrak{A}$. Therefore by definition ACF has algebraically prime models. ■

Corollary 3.20. By *Theorem 3.17*, ACF admits QE. ■

Remark 3.21. The proof RCF admits QE is quite similar, though the order needs to be taken account of at various stages. $\mathrm{RCF}_\forall$ is the theory of *ordered* integral domains and in a similar proof to *Lemma 3.19* RCF has algebraically prime models. Quantifier-free L_{or}-formulae are equivalent to

$$\bigwedge_{i=1}^{n} p_i(x) = 0 \wedge \bigwedge_{j=1}^{m} q_j(x) > 0$$

reminiscent of (1), and similarly to *Lemma 3.18* RCF is a simply closed theory. Thus RCF admits QE by *Theorem 3.17* as we wished to prove at the start of this section. ◊

Finally, we can define the concept of *model completeness*:

Definition 3.22. A theory T is called *model complete* if for all models $\mathfrak{M}, \mathfrak{N}$ of T,

$$\mathfrak{M} \subseteq \mathfrak{N} \Rightarrow \mathfrak{M} \prec \mathfrak{N}.$$

Marker [20] states this definition in a more elegant fashion: "T is model complete if and only if all embedding are elementary".

Remark 3.23. If T admits QE, T is model complete. This result follows from the fact all quantifier-free formulae are preserved passing to substructures and extensions. ◊

Corollary 3.24. *RCF and ACF are examples of model complete theories.* ■

3.2. The Tarski-Seidenberg Theorem and Hilbert's 17th Problem

Consider the following example of QE in action with applications to real algebraic geometry.

Example 3.25. Let $\mathfrak{R} = (\mathbb{R}, <, +, \times, 0, 1)$. By design $\mathfrak{R} \models \text{RCF}$ and thus admits QE, so we need only concern ourselves with Boolean combinations of the atomic formulae. In one variable the atomic formulae are given by

$$\sum_{i=0}^{n} a_i x^i = 0 \qquad \text{or} \qquad \sum_{i=0}^{n} a_i x^i < 0.$$

In general Boolean combinations of the atomic formulae define the semialgebraic sets (*Definition 1.2*). ♢

Thus $\mathfrak{R}$ is a structure whose definable sets are the semialgebraic sets. Any semialgebraic set is also definable in this structure. We can immediately deduce the *Tarski-Seidenberg theorem*:

Theorem 3.26. The Tarski-Seidenberg theorem. *The semialgebraic sets are closed under projection.*

Proof.
Let $A \subseteq \mathbb{R}^n$ be a semialgebraic set. Suppose A is defined in $\mathfrak{R}$ by $\varphi_A(x_1, \ldots, x_n)$. The set

$$B := \{(x_1, \ldots, x_{n-1}) : \mathfrak{R} \models \exists x_n \varphi_A(x_1, \ldots, x_n)\}$$

is definable, and moreover is the projection of A onto $\mathbb{R}^{n-1}$. As B is definable, it is semialgebraic. Therefore the semialgebraic sets are closed under projection as required. ■

The semialgebraic sets are the definable sets for $\mathfrak{R}$, an o-minimal structure, therefore the study of o-minimal structures is a generalisation of the study of real semialgebraic geometry!

As another example of the power model theory can wield in algebra, we can easily prove the closure of a semialgebraic set is semialgebraic. We will restrict our attention once more to RCF and proceed as follows:

Lemma 3.27. *Let $A \subseteq \mathbb{R}^n$ be semialgebraic. The closure of A, $\text{cl}(A)$, is semialgebraic.*

Proof.
Let $\mathfrak{R} \models \text{RCF}$. By the above comments there is a one-to-one correspondence between the semialgebraic sets and the definable sets of $\mathfrak{R}$. A is therefore defined by some $\varphi_A(x_1, \ldots, x_n)$. Let d be a function defined by

$$d(x_1, \ldots, x_n, y_1, \ldots, y_n) = z \quad \Leftrightarrow \quad (z \geq 0 \wedge (z^2 = \sum_{i=1}^{n} (x_i - y_i)^2)).$$

In this way, we get a 'metric' on our space, which is definable. Then

$$\mathrm{cl}(A) = \{x : \forall \epsilon > 0 \ \exists y \in A \text{ s.t. } d(x, y) < \epsilon\}$$

is definable, by

$$\mathrm{cl}(A) = \{(x_1, \ldots, x_n) : \forall \epsilon > 0 \ \exists y_1, \ldots, y_n \text{ s.t. } \varphi_A(y_1, \ldots, y_n) \wedge d(x_1, \ldots, x_n, y_1, \ldots, y_n) < \epsilon\}.$$

$\mathrm{cl}(A)$ is therefore semialgebraic, as required. ■

Remark 3.28. The fact that the interior of a semialgebraic set is semialgebraic follows without too much difficulty in a similar way. ♢

We can also give a very short proof to what was at one time one of the most difficult questions in mathematics, concerning *rational functions*:

Definition 3.29. Let A be a commutative ring and $K(x_1, \ldots, x_n)$ the field of fractions of $A[x_1, \ldots, x_n]$. An element $f(x_1, \ldots, x_n) \in K(x_1, \ldots, x_n)$ is called a *rational function*. Moreover, it can be written

$$f(x_1, \ldots, x_n) = \frac{g(x_1, \ldots, x_n)}{h(x_1, \ldots, x_n)} \qquad \text{for } f, g \in A[x_1, \ldots, x_n].$$

Theorem 3.30. Hilbert's 17th Problem. *If f is a multivariate rational polynomial taking only non-negative values over $\mathbb{R}$, then f is a sum of squares of rational functions.*

Proof.
Let $F \models$ RCF and suppose $f(x_1, \ldots, x_n) \in F(x_1, \ldots, x_n)$ is a rational function taking only non-negative values over F such that f is *not* a sum of squares. Note that $F(x_1, \ldots, x_n)$ is formally real, thus by *Theorem 3.4* $F(x_1, \ldots, x_n)$ is orderable. Moreover as $f(x_1, \ldots, x_n)$ is not a sum of squares, there is an ordering making $-f(x_1, \ldots, x_n)$ positive, i.e. making $f(x_1, \ldots, x_n)$ negative.

Let $\mathfrak{R}$ be the real closure of $F(x_1, \ldots, x_n)$ (extending the order on $F(x_1, \ldots, x_n)$ making $f(x)$ negative). Then $\mathfrak{R} \models \exists x(f(x) < 0)$. Since $\mathfrak{R} \models$ RCF and $F \subseteq \mathfrak{R}$, by the model completeness of RCF (*Corollary 3.24*) we have $F \models \exists x(f(x) < 0)$ by elementary equivalence. We have obtained a contradiction to the fact that f takes only non-negative values of F.

Therefore, f is a sum of squares as required. ■

3.3. Hilbert's Nullstellensatz

We will now restrict our attention to ACF where quantifier elimination has a geometric interpretation. The model theory facts presented are drawn from [20] and the algebra facts from [14].

3.3.1. ACF and Algebraic Geometry

Definition 3.31. Let K be a field and $S \subseteq K[x_1, \ldots, x_n]$ be an ideal. Define:

$$V(S) := \{a \in K^n : p(a) = 0 \ \forall p \in S\}.$$

When $K = \mathbb{R}$ or $\mathbb{C}$, this set is known as an *algebraic variety.* If $S = \{p\}$ we write $V(p) = V(\{p\})$.

Lemma 3.32. *Let S be the ideal generated by $p \in K[x_1, \ldots, x_n]$.*

$$V(p) = V(S).$$

■

Definition 3.33. Let $Y \subseteq K^n$. Then

$$\mathcal{I}(Y) := \{f \in K[x_1, \ldots, x_n] : \forall a \in Y,\ f(a) = 0\}.$$

Finally, to introduce topology we can define:

Definition 3.34. $X \subseteq K^n$ is *Zariski closed* if $X = V(S)$ for some ideal $S \subseteq K[x_1, \ldots, x_n]$.

The name “Zariski closed” for these sets is apt: it can be proven the Zariski closed sets form the closed sets of a topology.

With this definition in mind we can give geometric properties of the definable subsets of algebraically closed fields:

Lemma 3.35. *Let K be a field (not necessarily algebraically closed). A subset of K^n is quantifier-free definable if and only if it is a Boolean combination of Zariski closed sets.*

Proof.
See Marker [20, Lemma 3.2.7]. ■

We immediately deduce the following result:

Theorem 3.36. *If K is an algebraically closed field and $X \subseteq K$ is definable, then X is finite or cofinite.*

Proof.
Since $K \models$ ACF, by QE X is quantifier-free definable. By *Lemma 3.35* X is thus a finite Boolean combination of sets of the form $V(S)$ for some ideal $S \subseteq K[x_1, \ldots, x_n]$. If $S = \{0\}$ then $V(S) = K$; otherwise, $V(S)$ is finite. Therefore, X or $K \setminus X$ is finite as required. ■

We can relate this theorem back to ACF and model-theoretic results:

Definition 3.37. An L-theory T is *strongly minimal* if for all models $\mathfrak{A}$ of T, every definable subset of $\mathfrak{A}$ is finite or cofinite.

Corollary 3.38. *ACF is strongly minimal.* ■

3.3.2. Chevalley's Theorem and Hilbert's Nullstellensatz

Lemma 3.35 has another geometric implication known as *Chevalley's theorem.* To prove *Chevalley's theorem* (and later the *Nullstellensatz*), we will closely follow Marker's exposition [20, §3.2].

Definition 3.39. Let K be a field. A set $X \subseteq K^n$ is *constructible* if it is a finite Boolean combination of Zariski closed sets.

Theorem 3.40. *Let K be an algebraically closed field.*

(1) *$X \subseteq K^n$ is constructible if and only if it is definable.*

(2) **Chevalley's theorem.** *The image of a constructible set under a polynomial map is constructible.*

Proof.
(1) follows immediately from QE and *Lemma 3.35.*
If X is constructable and $p : K^n \to K^m$ a polynomial map, then the image of X under p,

$$p(X) = \{y \in K^m : \exists x \in K^n \text{ s.t. } p(x) = y\}$$

is definable, thus quantifier-free definable by QE, therefore constructible by *Lemma 3.35* as required. ■

Finally, using the model completeness of ACF and some algebra results, *Hilbert's Nullstellensatz* can be proven. The proof relies heavily on *Noetherian rings* and *Hilbert's Basis theorem* (which we will use without proof).

Definition 3.41. A commutative unital ring R is called *Noetherian* if every ideal I of R is finitely generated.

Theorem 3.42. Hilbert's Basis theorem. *Let R be a Noetherian ring. Then the polynomial ring $R[x_1, \dots, x_n]$ is also Noetherian.* ■

Definition 3.43. Let R be a commutative ring. The *radical* of an ideal I is the set of elements

$$\sqrt{I} := \{a \in R : \text{For some positive integer } n,\ a^n \in I\}$$

$\sqrt{I}$ is itself an ideal, and if $\sqrt{I} = I$ then I is known as a *radical ideal.*

The *Nullstellensatz* arose from the following question:

> Let K be a field and $I \subseteq K[x_1, \dots, x_n]$ an ideal. Consider the ideal
>
> $$\mathcal{I}(V(I)) := \{f \in K[x_1, \dots, x_n] : \forall a \in V(I),\ f(a) = 0\}$$
>
> We know $I \subseteq \mathcal{I}(V(I))$. When is $I = \mathcal{I}(V(I))$?

Proposition 3.44. *Let K be an algebraically closed field. If $I \subseteq K[x_1, \ldots, x_n]$ is a* radical ideal*, then $I = \mathcal{I}(V(I))$.*

To prove this proposition, we will first define a *prime* ideal:

Definition 3.45. Let R be a commutative ring. A *prime* ideal I is a proper ideal of R such that

$$\forall x, y \in R,\ xy \in I \Rightarrow x \in I \text{ or } y \in I.$$

We will also cite the following result without proof (obtained from Marker [20, Lemma 3.2.10]).

Lemma 3.46. Primary Decomposition. *Let K be a field. If $I \subsetneq K[x_1, \ldots, x_n]$ is a radical ideal, then there is a unique collection of prime ideals $Q_1, \ldots, Q_r$ containing I such that $I = Q_1 \cap \cdots \cap Q_r$.* ■

With this lemma in mind, we can give a proof of the *Nullstellensatz* relying on the model completeness of algebraically closed fields.

Theorem 3.47. Hilbert's Nullstellensatz. *Let $K \models$ ACF. If I, J are radical ideals of $K[x_1, \ldots, x_n]$ with $I \subsetneq J$, then $V(J) \subsetneq V(I)$.*

Proof.
Let $p \in J \setminus I$. Consider the primary decomposition $\{Q_1, \ldots, Q_r\}$ of I given by *Lemma 3.46*; there is thus a prime ideal Q containing I such that $p \notin Q$.

As Q is prime, it follows $K[x_1, \ldots, x_n]/Q$ is an integral domain. We denote by F the algebraic closure of the fraction field of $K[x_1, \ldots, x_n]/Q$.

By *Hilbert's Basis theorem* (*Theorem 3.42*) we know $K[x_1, \ldots, x_n]$ is Noetherian. Suppose $q_1, \ldots, q_m \in K[x_1, \ldots, x_n]$ generate I. Let $a_i \in F$ be the element $x_i + Q = x_i/Q$ in $K[x_1, \ldots, x_n]/Q$. Let $a = (a_1, \ldots, a_n)$. Note the following:

- $\forall i$, $q_i \in Q$ so $q_i(a) = q_i(x) + Q = 0$ in F.
- $p \notin Q$ so $p(a) = p(x) + Q \neq 0$ in F.

Therefore

$$F \models \left(\bigwedge_{i=1}^{m} q_i(a) = 0 \right) \wedge (p(a) \neq 0).$$

Thus

$$F \models \exists x \left(\bigwedge_{i=1}^{m} q_i(x) = 0 \right) \wedge (p(x) \neq 0)$$

and by the model-completeness of ACF,

$$K \models \exists x \left(\bigwedge_{i=1}^{m} q_i(x) = 0 \right) \wedge (p(x) \neq 0).$$

There is thus an element in K satisfying this: there exists $b \in K$ such that $q_i(b) = 0$ for all $i = 1, 2, \ldots$ and $p(b) \neq 0$. Therefore, there is an element b in $V(I) \setminus V(J)$, which means $V(J) \subsetneq V(I)$ as required. ■

To prove *Proposition 3.44*, it is now only a matter of applying definitions and noticing a contradiction:

Corollary 3.48. *Let K be an algebraically closed field. If $I \subseteq K[x_1, \ldots, x_n]$ is a radical ideal, then $I = \mathcal{I}(V(I))$.*

Proof.
It is immediate that $I \subseteq \mathcal{I}(V(I))$. If $I \subsetneq \mathcal{I}(V(I))$, by the *Nullstellensatz*

$$V(\mathcal{I}(V(I))) \subsetneq V(I). \tag{2}$$

For the sake of clarity, let $S = V(I)$. If $a \in S$ then for all $f \in \mathcal{I}(S)$, $f(a) = 0$, so $a \in V(\mathcal{I}(S))$.

We have just shown $S \subseteq V(\mathcal{I}(S))$, meaning

$$V(I) \subseteq V(\mathcal{I}(V(I))).$$

This is a contradiction to (2), so $I = \mathcal{I}(V(I))$ as required. ■

We can conclude that quantifier elimination is a powerful tool to wield in abstract algebra. Now the question becomes *what if our structure does not admit quantifier elimination?* As we shall see, o-minimality is an easier to satisfy condition and yet is still a powerful tool in itself.

4. O-Minimal Structures

Much of the following material is obtained from van den Dries [6], Marker [20] and Macpherson [17], the last being a concise survey on o-minimal structures that presents many ideas from [6] in a model-theoretic context.

First, we give a collection of definitions which form the foundation of this topic:

Definition 4.1. Let $\mathfrak{A}$ be an L-structure. A set $X \subseteq |\mathfrak{A}|^n$ is said to be *definable with parameters* or *parametrically definable* if there is an L-formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_k)$ and $b_1, \dots, b_k \in \mathfrak{A}$ (the *parameters*) such that

$$X = \{(a_1, \dots, a_n) : \mathfrak{A} \models \varphi(a_1, \dots, a_n, b_1, \dots, b_k)\}.$$

Definition 4.2. An *(open) interval* I in an L-structure $\mathfrak{A}$ is a parametrically definable subset of $\mathfrak{A}$ of the form

$$I = \{c : \mathfrak{A} \models \varphi(c, a, b)\}$$

where $\varphi(x, a, b) = a < x < b$ and $a, b \in |\mathfrak{A}| \cup \{\infty, -\infty\}$ with $a < b$.

Definition 4.3. An L-structure $\mathfrak{A}$ is said to be *o-minimal* if every parametrically definable subset of $\mathfrak{A}$ is a finite union of singletons and open intervals.

This relatively small and unassuming definition has a tremendous impact on the efforts to develop a general model theory encompassing all ordered structures. For model theorists there is a strong correlation between ease and analysis where any heavily analysed structure must be 'nice' in some regard in order for it to be well understood. We will see in the coming sections that o-minimality is a weak condition (thus is applicable in many problems) but has surprising power and the ability to be well understood.

Remark 4.4. In a language including $<$, we can define the relation "$\leq$" as follows:

$$\forall x, y \ (x \leq y \leftrightarrow (x < y \vee x = y))$$

Consider the following collection of sentences defining *linear order*, *dense*, and *without endpoints* in first order logic:

$$\textit{Linear order} \begin{cases} \forall a, b \ (a \leq b \vee b \leq a) \\ \forall a, b \ ((a \leq b \wedge b \leq a) \rightarrow (a = b)) \\ \forall a, b, c \ ((a \leq b \wedge b \leq c) \rightarrow (a \leq c)) \end{cases} \tag{3}$$

$$\textit{Dense} \begin{cases} \forall a, b \ (a < b \rightarrow (\exists c \ (a < c \wedge c < b)) \end{cases} \tag{4}$$

$$\textit{Without endpoints} \begin{cases} \forall a \ \exists b \ (b < a) \\ \forall a \ \exists b \ (a < b) \end{cases} \tag{5}$$

Let LO denote collection (3): the theory of linear orders.
Let DLO denote collections (3) & (4): the theory of dense linear orders.

Let DLO\E denote collections (3), (4) & (5): the theory of dense linear orders without endpoints.

From this point on, we will assume all languages L contain $<$ and all L-structures $\mathfrak{A} \models$DLO\E. $\mathfrak{A}$ with $<$ is known as a *dense linear order without endpoints.* $\Diamond$

Remark 4.5. O-minimality can be considered as a weaker form of quantifier elimination; an L-structure $\mathfrak{A}$ is o-minimal if and only if every L-formula in one free variable with parameters in $|\mathfrak{A}|$ is equivalent to a quantifier-free formula involving $<$ also with parameters in $|\mathfrak{A}|$ [26]. Thus, if a structure is o-minimal, it might not have all of the benefits and simplicity associated with quantifier elimination, but there will be some order and tameness imposed as demonstrated in §8. $\Diamond$

4.1. Examples of O-Minimal Structures

Example 4.6. Let $\mathfrak{A} = (\mathbb{Q}, <)$. The formulae of this language can only involve $<$ and $=$ meaning the 'solution set' to any single variable formula with parameters can only be a union of intervals and points. This means any parametrically definable subsets of $\mathfrak{A}$ are necessarily a finite union of singletons and open intervals, which makes $\mathfrak{A}$ o-minimal.
For example, if φ is the formula

$$\varphi(x, a, b, c) = (a < x < b) \vee (x = c)$$

then the solution set is $(a, b) \cup \{c\}$.
If we expand the language and set $\mathfrak{B} = (\mathbb{Q}, <, +)$, then the structure remains o-minimal. All that has changed is that the parameters (or parameters and variables) can now involve sums. $\Diamond$

To clear a point of possible confusion the following example is offered:

Example 4.7. Let $\mathfrak{A} = (\mathbb{Q}, <)$ and consider the formula

$$\varphi(x_1, x_2, a, b) = (a < x_1 < b) \wedge (x_2 = x_2)$$

This defines the set $(a, b) \times \mathbb{Q} \subseteq \mathbb{Q}^2$, which is not a finite union of intervals and points. However, *Definition 4.3* specifies that only parametrically definable **subsets of** $\mathfrak{A}$ need to be of this form.

$\mathfrak{A}$ is still o-minimal as φ is a formula defining a two dimensional set. $\Diamond$

Example 4.8. Coming from *Remark 4.5*, let $\mathfrak{R} = (\mathbb{R}, <, +,, \times, 0, 1)$ be a model of RCF. In §3 we proved this structure admits quantifier elimination so we just check that the single variable atomic formulae with parameters have solution sets consisting of finite unions of intervals and singletons. This point becomes clear when we write such formulae:

$$\varphi(x) = \qquad \sum_{i=0}^{n} a_i x^i = 0 \qquad \text{or} \qquad \sum_{i=0}^{n} a_i x^i < 0.$$

The solution sets of these formulae are indeed finite unions of intervals and points, which means $\mathfrak{R}$ is o-minimal as required. ♢

Example 4.9. Let $\mathfrak{R}_{\text{exp}} = (\mathbb{R}, <, 0, 1, +, -, \exp)$ be a model of the theory of real ordered fields with exponentiation. Wilkie [27] proved the following unprecedented result:

Theorem 4.10. Wilkie's theorem. *Let $\varphi(x_1, \dots, x_n)$ be a formula in the language L_r. There exists $m \geq n$ and $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_m, e^{x_1}, \dots, e^{x_m}]$ such that $\varphi(x_1, \dots, x_n)$ is equivalent to an existential formula:*

$$\exists x_{n+1}, \dots, x_m \left(f_1(x_1, \dots, x_m, e^{x_1}, \dots, e^{x_m}) = \cdots = f_s(x_1, \dots, x_m, e^{x_1}, \dots, e^{x_m}) = 0\right).$$

■

This result was the final piece to the puzzle initiated by Khovanski [12] with the following theorem:

Theorem 4.11. Khovanski's theorem. *Let $f_1, \dots f_s : \mathbb{R}^m \to \mathbb{R}$ be exponential polynomials.*[3] *Then*

$$Y = \{x \in \mathbb{R}^m : f_1(x) = \cdots = f_s(x) = 0\}$$

has finitely many connected components. ■

If we combine these two results, we obtain the o-minimality of $\mathfrak{R}_{\text{exp}}$. If $X \subseteq |\mathfrak{R}_{\text{exp}}|$ is definable by some $\varphi(x)$ then it is equivalent to some formula à la *Theorem 4.10*. X is then the *projection* of Y_φ given by *Theorem 4.10* and *Theorem 4.11* in $m \geq 1$ dimensions. However, Y_φ is a finite union of its connected components, meaning X is a finite union of points and intervals.

We should note that $\mathfrak{R}_{\text{exp}}$ does not admit quantifier elimination, but it is still o-minimal. [18, 28] discuss further this example and its history. ♢

Remark 4.12. More generally, we may wish to prove results instead about an o-minimal *expansion* of a theory. For example, $\mathfrak{M} = (\mathbb{R}, <, 0, 1, +, \times, -, \exp)$ is an expansion of the ordered field of real numbers (by which we mean a structure $(\mathbb{R}, <, 0, 1, +, \times, \dots)$ with additional symbols like "−","exp", to the usual symbols $\{<, 0, 1, +, \times\}$ of this theory). ♢

4.2. The Algebraic Approach to O-Minimality

Although o-minimality surfaced in model theory, for the purposes of algebraic geometry a deep understanding of the fundamental results in model theory is not needed. Recall the definition of an *o-minimal VDD structure* presented in §1:

[3] Polynomials including the exponential function.

Definition 4.13.

(1) A *VDD structure* on a nonempty set R is a sequence $\mathfrak{S} = (S_m)_{m\in\mathbb{N}}$ such that for each $m \geq 0$:

  (a) S_m is a Boolean algebra of subsets of R^m.

  (b) If $A \in S_m$ then $R \times A$, $A \times R \in S_{m+1}$.

  (c) $\{(x_1, \ldots, x_m) \in R^m : x_1 = x_m\} \in S_m$.[4]

  (d) If $A \in S_{m+1}$ then $\pi(A) \in S_m$ where $\pi : R^{m+1} \to R^m$ is the projection map to the first m coordinates.

(2) Let $(R, <)$ be a dense linearly ordered set without endpoints. A VDD structure $\mathfrak{S}$ on $(R, <)$ is called *o-minimal* if it is a VDD structure on R satisfying the additional conditions:

  (a) The sets in S_1 are exactly the finite unions of intervals and points.

  (b) $\{(x, y) \in R^2 : x < y\} \in S_2$ (that is, the relation $<$ belongs to S_2).

Remarks on Definition 4.13.

- A set $A \subseteq R^m$ is said to *belong* to $\mathfrak{S}$ if it is an element of some S_m. If $A \in S_m$ then A is said to be *definable.*
- A function $f : A \to B$ with $A \subseteq R^m$, $B \subseteq R^n$ is said to belong to $\mathfrak{S}$ if its graph

$$\Gamma(f) := \{(x, f(x)) : x \in A\} \qquad \subseteq R^{m+n}$$

  belongs to S_{m+n}. Thus a function is *definable* if its graph is.

$\Diamond$

Having two definitions of the term 'structure' is somewhat awkward, however the two definitions are equivalent. We will show the equivalence in the following theorem:

Theorem 4.14. Equivalence theorem. *Every o-minimal model-theoretic structure $\mathfrak{A}$ corresponds to an o-minimal VDD structure $\mathfrak{S}_{\mathfrak{A}} = (S_m)_{m\in\mathbb{N}}$, and every o-minimal VDD structure $\mathfrak{S}$ corresponds to an o-minimal model-theoretic structure $\mathfrak{A}_{\mathfrak{S}}$.*

Proof.
($\Rightarrow$). We wish to prove the definable sets of an o-minimal L-structure form an o-minimal VDD structure.
Suppose L contains $<$, and let $\mathfrak{A}$ be an o-minimal L-structure with domain $\mathbb{A}$. Denote the definable sets of $\mathfrak{A}$ by $Def(\mathfrak{A})$. Moreover, let $Def(\mathfrak{A})_m$ denote the definable subsets of $\mathbb{A}^m$. If $A \in Def(\mathfrak{A})_m$, let φ_A be the formula defining it. For all m:

- $\emptyset \in Def(\mathfrak{A})_m$, defined by $\varphi(x) =$ "$x \neq x$".

[4] *Lemma 4.15* will make this condition more general.

- If $A \in Def(\mathfrak{A})_m$, then $A^c \in Def(\mathfrak{A})_m$, defined by $\neg\varphi_A$.
- If $A, B \in Def(\mathfrak{A})_m$ then $A \cup B \in Def(\mathfrak{A})_m$, defined by $\varphi_A \vee \varphi_B$.

Thus $Def(\mathfrak{A})_m$ is a Boolean algebra. If $A \in Def(\mathfrak{A})_m$ then $\mathbb{A} \times A, A \times \mathbb{A} \in Def(\mathfrak{A})_m$ defined by

$$\phi(y, x) = \varphi_A(x) \wedge (y = y) \qquad \text{and} \qquad \phi(x, y) = \varphi_A(x) \wedge (y = y)$$

respectively. The set $\{(x_1, \ldots, x_m) \in \mathbb{A}^m : x_1 = x_m\}$ is immediately definable. Finally, if $A \in Def(\mathfrak{A})_{m+1}$ then $\pi(A) \in Def(\mathfrak{A})_m$ is defined by

$$\varphi_{\pi(A)}(x_1, \ldots, x_m) = \exists x_{m+1}\varphi_A(x_1, \ldots, x_{m+1}).$$

Note the correspondence between the projection map π and the existential quantifier $\exists$. This correspondence will come in useful later.

Note $\{(x, y) : x < y\} \in Def(\mathfrak{A})_2$, and by definition the sets of $Def(\mathfrak{A})_1$ are finite unions of points and intervals. Therefore, we can conclude $\mathfrak{S}_{\mathfrak{A}} := Def(\mathfrak{A})$ is an o-minimal VDD structure, as required.

Before we show the reverse direction, we will first cite a lemma that will be crucial to our proof:

Lemma 4.15. Switching lemma. [6, Chapter 1, Lemma 2.2]. *Let $\mathfrak{S}$ be a VDD structure on $\mathbb{A}$.*

(1) $\{(x_1, \ldots, x_n) : x_i = x_j\} \in \mathfrak{S}$.

(2) *If $B \in S_m$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$, then $A \subseteq \mathbb{A}^n$ defined by*

$$(x_1, \ldots, x_n) \in A \Leftrightarrow (x_{i_1}, \ldots, x_{i_m}) \in B$$

belongs to $\mathfrak{S}$.

Returning to the proof of Theorem 4.14:
($\Leftarrow$). The following proof originates in private correspondence with L. van den Dries.

Suppose $\mathfrak{S} = (S_m)_{m \in \mathbb{N}}$ is an o-minimal VDD structure on $\mathbb{A}$. Let L be the language $L = \{(R_X)_{X \in S_n, n \in \mathbb{N}}\}$ containing an n-ary relation symbol R_X for each $X \in S_n$, which is interpreted as the set X, i.e.

$$R_X(x) \quad \Leftrightarrow \quad x \in X \subseteq \mathbb{A}^n.$$

Let $\mathfrak{A}_{\mathfrak{S}}$ be an L-structure (with domain $\mathbb{A}$). We wish to prove the definable sets of this structure correspond to the definable sets in $\mathfrak{S}$, and we will do so by *induction on the complexity of formulae.*

Consider sets defined by atomic formulae. These are given by $R_X(x_1, \ldots, x_n)$ or "$x_i = x_j$". The latter defines a set in $\mathfrak{S}$ by *Lemma 4.15.* The former possibly needs *Lemma 4.15* if variables are not in order, or have been repeated. To see this, consider the following example:

Example 4.16. Consider $X \in S_3$ defined by the predicate R_X. $R_X(x_2, x_1, x_2)$ defines a set in $\mathbb{A}^2$, but is this set in S_2? Rewrite:

$$\begin{aligned} R_X(x_2, x_1, x_2) &= \exists y_1, y_2, y_3 (y_1 = x_2 \wedge y_2 = x_1 \wedge y_3 = x_2 \wedge R_X(y_1, y_2, y_3)) \\ &= \exists y_1, y_2, y_3 (\varphi(x_1, x_2, y_1, y_2, y_3)) \end{aligned}$$

$R_X(y_1, y_2, y_3)$ viewed as the formula $\varphi(x_1, x_2, y_1, y_2, y_3)$, a condition on $\mathbb{A}^2 \times \mathbb{A}^3$, defines the set $\mathbb{A}^2 \times X$. Moreover, the formula $x_i = x_j$ defines a set in $\mathfrak{S}$ by *Lemma 4.15*, meaning that

$$\varphi(x_1, x_2, y_1, y_2, y_3) \quad = \quad y_1 = x_2 \wedge y_2 = x_1 \wedge y_3 = x_2 \wedge R_X(y_1, y_2, y_3)$$

defines a set in $\mathfrak{S}$. Applying the existential quantifier three times is equivalent to taking the projection map $\mathbb{A}^5 \to \mathbb{A}^2$, and since $\mathfrak{S}$ is closed under projection, $R_X(x_2, x_1, x_2)$ defines a set in S_2, as required.

If we call 'Y' the set defined by $R_X(x_2, x_1, x_2)$, we obtain the following relation:

> If $X \in S_3$ and $i_1 = 2, i_2 = 1, i_3 = 2$, then $Y \subseteq \mathbb{A}^2$ defined by
>
> $$(x_1, x_2) \in Y \quad \Leftrightarrow \quad (x_{i_1}, x_{i_2}, x_{i_3}) = (x_2, x_1, x_2) \in X$$
>
> belongs to $\mathfrak{S}$.

As we can see, this statement is precisely *Lemma 4.15*. $\Diamond$

Therefore, atomic formulae define sets in $\mathfrak{S}$.
If sets are given by Boolean combinations of atomic formulae ($\varphi \wedge \phi$, $\neg\varphi$, etc.), since S_m is a Boolean algebra these sets remain in S_m. More generally, for sets in $\mathfrak{S}$, if

$$\varphi(x_1, \dots, x_m) = R_{A_n}(x_1, \dots, x_n) \wedge R_{A_m}(x_1, \dots, x_m) \qquad n < m,$$

then the set defined by φ

$$A_\varphi = R_{A_n} \times \mathbb{A}^{m-n} \cap R_{A_m} \quad \in S_m.$$

The existential quantifier is managed by $(1), (d)$ of *Definition 4.13*. Finally, $(2), (b)$ of *Definition 4.13* ensures the relation $<$ is definable, while $(2), (a)$ ensures $\mathfrak{A}_\mathfrak{S}$ is o-minimal.

Therefore, $\mathfrak{A}_\mathfrak{S}$ is an o-minimal structure whose definable sets correspond to $\mathfrak{S}$, as required. ■

It will often be important for us to prove things in a model-theoretic context and apply them to a context with VDD structures or vice versa. For instance, $\mathfrak{R} = (\mathbb{R}, <, +, \times, 0, 1)$ admits quantifier elimination by *Example 3.25*, and hence is o-minimal, with the semialgebraic sets being the definable sets. $\mathfrak{R}$ corresponds to an o-minimal VDD structure $\mathfrak{S}_\mathfrak{R}$ by *Theorem 4.14*, which is closed under projection. Therefore, the semialgebraic sets are closed under projection, which is the *Tarski-Seidenberg theorem* exactly.

4.3. Ordered Algebraic Structures and O-Minimality

We spoke earlier about the 'niceness' of o-minimal structures and what order this feature can impart on the rest of the structure. As an example, we shall show that o-minimal ordered groups are abelian and divisible and that o-minimal ordered rings are real closed fields (*Definition 3.3*) as shown first by Pillay and Steinhorn in [23] and later van den Dries in [6].

Lemma 4.17. *Let* $L = \{<, *\}$ *and* $\mathfrak{A}$ *be an o-minimal ordered group in the language of* L*. Then the only definable subgroups of* $\mathfrak{A}$ *are* $\{1\}$ *and* $\mathfrak{A}$*.*

Proof.
Let G be a definable subgroup of $\mathfrak{A}$. If G is not convex, then there exists $g \in G$, $r \in |\mathfrak{A}| \setminus G$ with $1 < r < g$. We can create an ordered sequence $1 < r < g < r * g < g * g < r * g * g < \dots$ whose terms alternate in and out of the definable set G. This sequence contradicts the o-minimality of $\mathfrak{A}$, which requires G to be a *finite* union of points and intervals. Therefore, G is convex.
Assume $G \neq \{1\}$. Then if $s := \sup(G) > 1$, we have $(1, s) \subseteq G$. Either $s = +\infty$ (in which case by symmetry $G = \mathfrak{A}$ and we are done) or $s < +\infty$, in which case

$$g > 1 \Rightarrow g^{-1} < 1 \Rightarrow g^{-1} * s \in G \Rightarrow s \in G \Rightarrow s < g * s \in G$$

by the definition of the ordering. This is a contradiction to s being the supremum. ∎

Using the same setup as the above lemma, we prove:

Theorem 4.18. $(\mathfrak{A}, *)$ *as a group is abelian and divisible.*

Proof.
For all $a \in \mathfrak{A}$, the centraliser $C_a = \{s \in \mathfrak{A} : a * s = s * a\}$ is a nontrivial definable subgroup of $\mathfrak{A}$. Thus $C_a = \mathfrak{A}$ by *Lemma 4.17*, making $(\mathfrak{A}, *)$ abelian.
For each $n > 0$ the subgroup $\{x^n : x \in \mathfrak{A}\}$ is definable, hence equal to $\mathfrak{A}$, making $\mathfrak{A}$ divisible as required. ∎

We now extend this idea from ordered groups to ordered rings:

Theorem 4.19. *Let* $L = \{<, +, \times\}$ *and* $\mathfrak{A}$ *be an o-minimal ordered ring in the language of* L*. Then* $\mathfrak{A}$ *is a real closed field.*

Proof.
We will slowly work through the axioms of a real closed field all the while using properties induced by o-minimality.

For all $a \in \mathfrak{A}$, $a\mathfrak{A}$ is a definable subgroup of $(\mathfrak{A}, +)$, hence $a\mathfrak{A} = \mathfrak{A}$ for $a \neq 0$ by *Lemma 4.17*. Thus $\mathfrak{A}$ is an ordered division ring. In particular, $(\mathfrak{A}, \times)$ is a group (as every nonzero element has an inverse).

Define $\mathrm{Pos}(\mathfrak{A}) := \{a \in \mathfrak{A} : a > 0\} \subseteq \mathfrak{A}$. Viewed as a substructure of $\mathfrak{A}$, $\mathrm{Pos}(\mathfrak{A})$ is an ordered multiplicative group. Then $\mathrm{Pos}(\mathfrak{A})$ is abelian by *Theorem 4.18*. Note the centraliser C_a is a definable subgroup of $(\mathfrak{A}, \times)$. Thus for all $a \in \mathfrak{A}$, $C_a = \{1\}$ or $C_a = \mathfrak{A}$ by *Lemma 4.17*. In particular, for all $b \in \mathrm{Pos}(\mathfrak{A})$, $C_b = \mathfrak{A}$, i.e.

$$\forall a \in \mathfrak{A}, \forall b \in \mathrm{Pos}(\mathfrak{A}) \quad a * b = b * a.$$

We can draw the same conclusions about $\mathrm{Neg}(\mathfrak{A}) := \{a \in \mathfrak{A} : a < 0\}$:

$$\forall a \in \mathfrak{A}, \forall b \in \mathrm{Neg}(\mathfrak{A}) \quad a * b = b * a.$$

As for all $a \in \mathfrak{A}, a * 0 = 0 * a = 0$, we conclude $(\mathfrak{A}, \times)$ is indeed commutative, making $\mathfrak{A}$ an ordered field.

Finally, it can be shown a one variable polynomial $f(x) \in \mathfrak{A}[x]$ gives rise to a definable (see *Remark 4.20*) continuous function. A straightforward derivation then gives that the image of a definable and connected set under a definable and continuous function is definable and connected. Since an interval (a, b) is definable and connected (in the interval topology), $(f(a), f(b))$ is definable and connected. The Intermediate Value Property thus holds, meaning $\mathfrak{A}$ is a real closed field by *Theorem 3.6*. ■

Remark 4.20. Recall a function is called definable if its graph is definable. If f is a polynomial in one variable over R, $f(x) = \sum_{i=0}^{n} a_i x^i$. The set

$$A = \{(x, f(x)) : x \in R\} = \Gamma(f)$$

is parametrically definable, so polynomials give rise to definable functions. ◊

5. The Monotonicity and Cell Decomposition Theorems

We will work with an arbitrary but fixed o-minimal structure $\mathfrak{A}$. The Monotonicity theorem and Cell Decomposition theorem of this section were proven by Knight et al. [13] and presented with further explanation by van den Dries in [6].

5.1. Monotonicity

The *Monotonicity theorem* in essence states that the definable functions are (up to a finite number of points) well behaved: constant or strictly monotone and continuous.

Theorem 5.1. Monotonicity theorem. *Let $f : I \to |\mathfrak{A}|$ be a definable function on an interval $I = (a, b)$ in $\mathfrak{A}$. Then there are points $a = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = b$ such that on each subinterval (a_j, a_{j+1}), f is either constant or strictly monotone and continuous.*

Proof.
See van den Dries [6, Chapter 3, Theorem 1.2]. A sketch is presented, relying on the following lemma:

Lemma 5.2.

(1) *If $f : I \to |\mathfrak{A}|$ is a definable function on an interval $I = (a, b)$, then there exists a subinterval I' of I on which either f is constant or injective.*

(2) *If f is injective, it is strictly monotone on a subinterval of I.*

(3) *If f is strictly monotone, it is continuous on some subinterval of I.* ■

The proof of *Theorem 5.1* follows:
Define

$$X = \{x \in I : \text{On some subinterval of } I \text{ containing } x,\ f \text{ is either constant or strictly monotone and continuous. }\}.$$

As X is definable, $I \setminus X$ is finite; otherwise by o-minimality there would be a subinterval of $I \setminus X$ on which we could successively apply **(1)**, **(2)**, **(3)** obtaining a subinterval I' of $I \setminus X$ on which f is either constant or strictly monotone and continuous. Then $I' \subseteq X$, a contradiction.

Hence we have a subdivision $a = a_0 < a_1 < a_2 < \cdots < a_k < a_{k+1} = b$ of I such that each subinterval $(a_i, a_{i+1}) \subseteq X$. We can assume f is continuous on each such interval.
For each interval (a_i, a_{i+1}), we can split up the interval further into three cases:

(1) For all $x \in (a', b') \subseteq (a_i, a_{i+1})$, f is constant on a neighbourhood of x. Define

$$D_x = \{y \in (a', b') : x < y \text{ and } f \text{ is constant on } [x, y)\}.$$

This set is definable, nonempty, and its supremum is b': a lower supremum s means f is then constant on some neighbourhood of $s \in (a', b')$, a contradiction. Thus f is constant on all of (a', b').

(2) For all $x \in (a', b')$, f is strictly increasing on a neighbourhood of x: again define

$$D_x = \{y \in (a', b') : x < y \text{ and } f \text{ is strictly increasing on } [x, y)\}$$

and (by the same reasoning) conclude f is strictly increasing on (a', b').

(3) For all $x \in (a', b')$, f is strictly decreasing on a neighbourhood of x: this is handled as in (2).

When we split (a_i, a_{i+1}) into the three cases, we note an infinite split contradicts o-minimality again; as noted by Pillay and Steinhorn [23], the a_i are definable, making (a_i, a_{i+1}) a definable set, so an infinite split of (a_i, a_{i+1}) would result in writing a definable set as an infinite union of points and intervals.

Thus, including the (finitely many) extra points resulting from using (1)-(3), we conclude there are points $a = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = b$ such that on each subinterval (a_j, a_{j+1}), f is either constant or strictly monotone and continuous, as required. ■

This theorem has an important corollary: the existence of limits in these structures.

Corollary 5.3. *Let $f : (a, b) \to |\mathfrak{A}|$ be a definable function. For all $c \in (a, b)$ the limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist in $|\mathfrak{A}| \cup \{-\infty, +\infty\}$. Also the limits $\lim_{x \to b^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist in $|\mathfrak{A}| \cup \{-\infty, +\infty\}$.*

Sketch proof.
Let $c \in (a, b)$. The *Monotonicity theorem* in simpler terms can be rephrased as "a definable function is continuous up to a finite number of bad points". Therefore if c is one of these "bad points" we know to the left and right of c the function is well behaved, and so on a small enough interval about c, $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist in $|\mathfrak{A}| \cup \{-\infty, +\infty\}$. If c is not one of these "bad points" then by the continuity of f the limits automatically exist. The same argument can be made about a or b. For example, about a, f is continuous and constant or strictly monotone on (a_0, a_1), thus $\lim_{x \to a^+} f(x)$ exists in $|\mathfrak{A}| \cup \{-\infty, +\infty\}$. ■

Remark 5.4. Consider the *topologist's sine curve*, given by

$$T = \left\{ \left(x, \sin \frac{1}{x}\right) : x \in (0, 1] \right\} \cup \{(0, 0)\}.$$

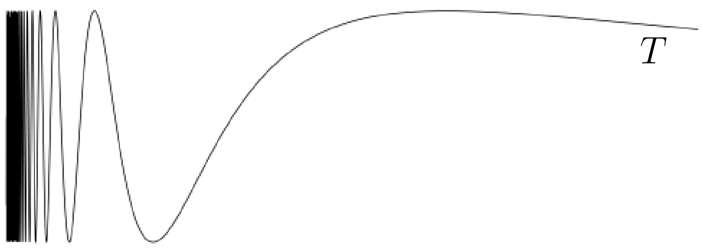


Figure 2: *Topologist's sine curve.* Image credit: `D. Dumas`.

This is a continuous function, and yet $\lim_{x\to 0^+} \sin\frac{1}{x}$ does not exist. Therefore, this function cannot be definable in an o-minimal structure. ♢

5.2. Cell Decomposition

We will show the definable subsets of $|\mathfrak{A}|^m$ split into nice collections, known as *cells*, and that each definable function on $|\mathfrak{A}|^m$ is cell-wise continuous. We will see for $m = 1$ that the decomposition is one into finitely many points and intervals: that is, we will see the cell decomposition respects the o-minimality of the structure. Also the functions on $|\mathfrak{A}|$ being cell-wise continuous will reduce to the *Monotonicity theorem* as one might expect.

In this section, we will always take π to be the usual projection map $\pi : |\mathfrak{A}|^{m+1} \to |\mathfrak{A}|^m$ unless otherwise stated, where m will be clear from the context.

Define the following notation: for X a definable set in $|\mathfrak{A}|^m$,

- $C(X) := \{f : X \to |\mathfrak{A}| : \ f \text{ is definable and continuous}\}$.
- $C_\infty(X) := C(X) \cup \{-\infty, \infty\}$ (where $\pm\infty$ are functions).
- For $f, g \in C_\infty(X)$, define
$$(f, g)_X := \{(x, a) \in X \times |\mathfrak{A}| : f(x) < a < g(x)\}$$
(making this a definable subset of $|\mathfrak{A}|^{m+1}$).

Definition 5.5. Let $(i_1, \dots, i_m) \in \{0, 1\}^m$, $m \geq 1$. An $(i_1, \dots, i_m)$-*cell* is a definable subset of $|\mathfrak{A}|^m$ obtained by induction as follows:

(1) A (0)-cell is a point $\{a\} \subseteq |\mathfrak{A}|$.

(2) A (1)-cell is an interval $(a, b) \subseteq |\mathfrak{A}|$.

(3) Given an $(i_1, \dots, i_m)$-cell an $(i_1, \dots, i_m, 0)$-cell is the graph of a function $f \in C(X)$ where X is an $(i_1, \dots, i_m)$-cell.

(4) Given an $(i_1, \dots, i_m)$-cell an $(i_1, \dots, i_m, 1)$-cell is a set $(f, g)_X$ where X is an $(i_1, \dots, i_m)$-cell and $f < g \in C_\infty(X)$.

Example 5.6. A $(1, 0)$-cell is the graph of a continuous function on an interval. A $(0, \dots, 0)$-cell is a point in $|\mathfrak{A}|^m$ where m is the size of the tuple $(0, \dots, 0)$. ♢

We acquire a sense of topological 'openness' with *Definition 5.5*: the $(1, \dots, 1)$-cells are known as *open cells* (and are open in the product topology on $|\mathfrak{A}|^m$).

If an $(i_1, \dots, i_m)$-cell has $i_j = 0$ for some j (i.e. the cell is not open), it is 'thin' in the sense that it has empty interior. Moreover, a union of finitely many non-open cells has empty interior. Informally, we see this result as any non-open cell is 'flat' in one direction (leading to an empty interior) and a finite union of such cells cannot fix this 'flatness'.

Remark 5.7. We can in fact map each cell homeomorphically under a coordinate projection to an *open* cell of lower dimension by the following:

Let $i \in \{0,1\}^m$, $i = (i_1, \ldots, i_m)$. Set $k = i_1 + \cdots + i_m$ and let $\lambda(1) < \cdots < \lambda(k)$ be indices, where $\lambda \in \{1, \ldots, m\}$ and $i_\lambda = 1$ in[5] i. Define $p_i : |\mathfrak{A}|^m \to |\mathfrak{A}|^k$ by

$$p_i(x_1, \ldots, x_m) = (x_{\lambda(1)}, \ldots, x_{\lambda(k)}).$$

We are essentially discarding the "0's" from $i = (i_1, \ldots, i_m)$, which has little consequence as the cell is 'flat' in this direction. ♢

We now speak about a particular partition of $|\mathfrak{A}|^m$ into finitely many cells known as a *decomposition*:

Definition 5.8. We define by induction:

(1) A *decomposition* of $|\mathfrak{A}|^1 = |\mathfrak{A}|$ is a collection of intervals and points

$$\{(-\infty, a_1), (a_2, a_3), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$$

where $a_1 < \cdots < a_k$ in $|\mathfrak{A}|$.

(2) A decomposition of $|\mathfrak{A}|^{m+1}$ is a finite partition of $|\mathfrak{A}|^{m+1}$ into cells C such that the set of projections $\{\pi(C)\}$ is a decomposition of $|\mathfrak{A}|^m$.

If we have a decomposition $\mathbb{D}$ of $|\mathfrak{A}|^m$, we can obtain a decomposition $\mathbb{D}^*$ of $|\mathfrak{A}|^{m+1}$ by the following method:

Label $\mathbb{D} = \{A_1, \ldots A_k\}$ with A_i the (distinct) cells of the decomposition and suppose for each $1 \leq i \leq k$ we are given functions $f_{j_1} < \cdots < f_{j_{n_i}}$ in $C(A_i)$. Then

$$D_i := \{(-\infty, f_{j_1}), (f_{j_2}, f_{j_3}), \ldots, (f_{j_{n_i}}, +\infty), \Gamma(f_{j_1}), \ldots, \Gamma(f_{j_{n_i}})\}$$

partitions $A_i \times |\mathfrak{A}|$. Moreover, $\mathbb{D}^* = D_1 \cup \cdots \cup D_k$ is a decomposition of $|\mathfrak{A}|^{m+1}$ with $\mathbb{D} = \pi(\mathbb{D}^*)$ (*fig. 3*).

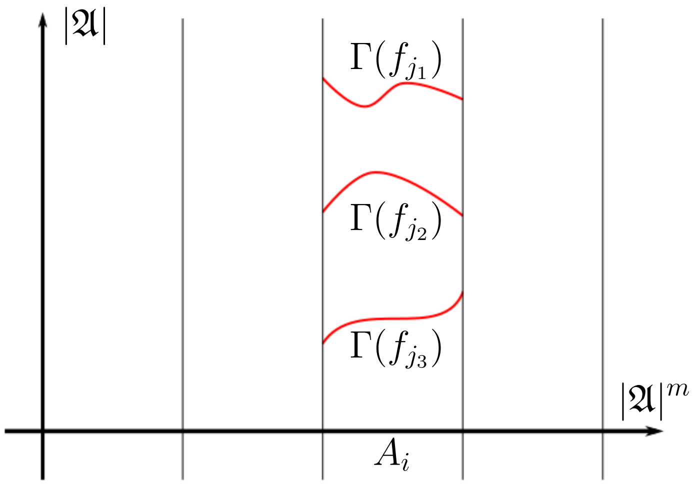


Figure 3

[5] The "1's" in $(i_1, \ldots, i_m)$.

Definition 5.9. Given a set $S \subseteq |\mathfrak{A}|^m$, a decomposition $\mathbb{D}$ of $|\mathfrak{A}|^m$ *partitions* S if S is a union of cells in $\mathbb{D}$.

Finally, if we want a measure of finiteness or boundedness of definable sets in higher dimensions, we turn to the *uniform finiteness property*, proven as part of *Theorem 5.11.* The definition of *uniformly finite* is as follows:

Definition 5.10.

- A set $Y \subseteq |\mathfrak{A}|^{m+1}$ is *finite over* $|\mathfrak{A}|^m$ if for each $x \in |\mathfrak{A}|^m$ the *fiber*

$$Y_x := \{a \in |\mathfrak{A}| : (x, a) \in Y\}$$

  is finite.
- Y is *uniformly finite over* $|\mathfrak{A}|^m$ if there exists a natural number N such that for all $x \in |\mathfrak{A}|^m$, $\|Y_x\| \leq N$.

Now we turn to one of the main results in o-minimality: the *Cell Decomposition theorem.*

Theorem 5.11. Cell Decomposition theorem. *For $m > 0$ the following hold:*

$(\mathrm{I})_\mathrm{m}$ *Given definable sets $A_1, \ldots, A_k \subseteq |\mathfrak{A}|^m$, there is a decomposition of $|\mathfrak{A}|^m$ partitioning each of the $A_1, \ldots, A_k$.*

$(\mathrm{II})_\mathrm{m}$ *Given a definable function $f : A \to |\mathfrak{A}|$, $A \subseteq |\mathfrak{A}|^m$, there is a decomposition of $|\mathfrak{A}|^m$ partitioning A such that the restriction $f|_B : B \to |\mathfrak{A}|$ to each cell B in the decomposition with $B \subseteq A$ is continuous.*

$(\mathrm{III})_\mathrm{m}$ *If a definable subset $Y \subseteq |\mathfrak{A}|^{m+1}$ is finite over $|\mathfrak{A}|^m$, then it is uniformly finite over $|\mathfrak{A}|^m$.*

■

Remark 5.12. In the *Cell Decomposition theorem*, $(\mathrm{I})_\mathrm{m}$ ensures the existence of a partition of a definable set by a cell decomposition, $(\mathrm{II})_\mathrm{m}$ is a generalisation of the *Monotonicity theorem*, and $(\mathrm{III})_\mathrm{m}$ is a generalisation of the same finiteness property but for semialgebraic sets ([20, Lemma 3.3.30]). ♢

The full proof of the *Cell Decomposition theorem* is given by Knight et al. [13] and van den Dries [6, Chapter 3, Theorem 2.11]. Note that property $(\mathrm{III})_\mathrm{m}$, the uniform finiteness property, is a property about definable sets (thus a property about $\mathrm{Th}(\mathfrak{A})$) and is preserved by elementary equivalence. This strong fact allows us in Macpherson's survey [17] to prove an appealing property of o-minimal structures: they are preserved under elementary equivalence.

Theorem 5.13. *Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. If $\mathfrak{A}$ is o-minimal and $\mathfrak{B} \equiv \mathfrak{A}$, then $\mathfrak{B}$ is o-minimal.*

Proof.
Let $X \subseteq |\mathfrak{B}|$ be definable by a formula $\varphi(x, b)$, and suppose the formula $\phi(x, b)$ defines the boundary of X. By the uniform finiteness property on $\mathfrak{A}$, there is a bound $N \in \mathbb{N}$ such that $\forall a \in |\mathfrak{A}|$, $\phi(x, a)$ has at most N realisations. As previously stated, this bound is a property of $\mathrm{Th}(\mathfrak{A})$, and thus by elementary equivalence it is also a property of $\mathrm{Th}(\mathfrak{B})$. So $\phi(x, b)$ has at most N realisations and thus X defined by $\varphi(x, b)$ is a union of finitely many singletons and intervals. We can conclude $\mathfrak{B}$ is o-minimal, as required. ■

Note that even though o-minimality can be regarded as 'weak' quantifier elimination, this theorem shows it still has considerable strength. O-minimality is not limited to a particular structure but instead is a property of theories like quantifier elimination.

Other applications of the *Cell Decomposition theorem* are immediate to the topology of the space:

Definition 5.14. A subset of $|\mathfrak{A}|^n$ (for any $n \in \mathbb{N}$) is *definably connected* if it is definable and connected.

Corollary 5.15. *Let $\mathbb{D}$ be a cell decomposition of $\mathfrak{A}$. Each cell in $\mathbb{D}$ is definably connected.* ■

Definition 5.16. A *definably connected component* of a nonempty definable set $X \subseteq |\mathfrak{A}|^m$ is a maximal definably connected subset of X.

Corollary 5.17. *If $X \subseteq |\mathfrak{A}|^m$ is a nonempty definable set it has only finitely many definably connected components. Moreover, the components are closed and open in X and form a finite partition of X.* ■

5.3. Fibers, Definable Families, and Trivialization

A useful fact about cells concerns their fibers, where in general a fiber of a set $Y \subseteq |\mathfrak{A}|^{m+n}$ is an n-dimensional object $Y_x \subseteq |\mathfrak{A}|^n$ for $x \in |\mathfrak{A}|^m$ as opposed to the one-dimensional fiber presented in *Definition 5.10.* Let $\pi : |\mathfrak{A}|^{m+n} \to |\mathfrak{A}|^m$ be the standard projection to the first m coordinates. Then:

Lemma 5.18.

(1) *Let C be a cell in $|\mathfrak{A}|^{m+n}$ and $a \in \pi(C)$. Then C_a is a cell in $|\mathfrak{A}|^n$.*

(2) *If $\mathbb{D}$ is a decomposition of $|\mathfrak{A}|^{m+n}$ and $a \in |\mathfrak{A}|^m$, then $\mathbb{D}_a$ given by*

$$\mathbb{D}_a := \{C_a : C \in D \wedge a \in \pi(C)\}$$

is a decomposition of $|\mathfrak{A}|^n$.

Remark 5.19. We know the projection of a decomposition to a lower dimension (by definition) is a decomposition. This lemma tells us the projection 'in the other direction' (that is, from an $m+n$ tuple remove the first m coordinates) also forms a decomposition. $\Diamond$

Sketch proof of Lemma 5.18:
(2) follows easily once we know the C_a are cells. Visually, we can think of C_a as a slice of C in the a^{th} direction. Once we picture this, we can 'compress' the decomposition of $|\mathfrak{A}|^{m+n}$ to a decomposition of $|\mathfrak{A}|^n$ by focusing on how the a-slice is divided. We see the a-slice is indeed decomposed by the C_a, as required (*fig. 4*).

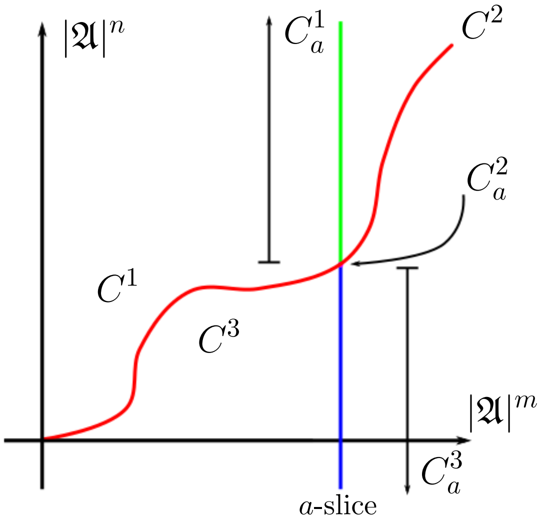


Figure 4

We now prove (1), namely that the C_a are cells. We prove it by induction on n, the trick being to use the projective property of cells at the induction step. Define

$$\Pi : |\mathfrak{A}|^{m+(n+1)} \to |\mathfrak{A}|^m \qquad \text{by} \qquad |\mathfrak{A}|^{m+(n+1)} \xrightarrow{\pi_1} |\mathfrak{A}|^{m+n} \xrightarrow{\pi} |\mathfrak{A}|^m,$$

where π_1 is the standard projection. Suppose C is a cell in $|\mathfrak{A}|^{m+n+1}$. Fix $a \in \Pi(C)$ and assume $(\pi_1 C)_a$ is a cell. We will show C_a is a cell in $|\mathfrak{A}|^{n+1}$. There are two cases:

- If $C = \Gamma(f)$ then defining $f_a : (\pi_1 C)_a \to |\mathfrak{A}|$ by $f_a(x) = f(a,x)$ we obtain $C_a = \Gamma(f_a)$ (as we are fixing a).
- Similarly, if $C = (f,g)_X$ then $C_a = (f_a, g_a)_{X_a}$.

The base case of $n = 1$ is clear. We obtain either a point or interval from C_a (that is, a (0)-or (1)-cell) from the definition of a fiber. $\blacksquare$

While in the context of fibers, we can define *definable families*:

Definition 5.20. Let $S \subseteq |\mathfrak{A}|^{m+n}$ be definable. For every $a \in |\mathfrak{A}|^m$, consider fibers S_a. The *definable family* $(S_a)_{a\in|\mathfrak{A}|^m}$ of subsets of $|\mathfrak{A}|^n$ parametrised by elements in $|\mathfrak{A}|^m$ is then described by S. The sets S_a of the definable family are known as the *fibers of the family.*

Example 5.21. Let $L = \{<, +, \times\}$, and let $\mathfrak{A}$ be an L-structure with domain $\mathbb{R}$. The equation

$$ax^2 + bx + c = 0 \tag{6}$$

defines a set $S \subseteq \mathbb{R}^3 \times \mathbb{R}$. The fibers $S_{(a,b,c)}$ of the family (with parameter space $\mathbb{R}^3$) are the solutions of (6), meaning $S_{(a,b,c)}$ can be the empty set, a single point, two points, or all of $\mathbb{R}$ depending on the values of a, b, c. ♢

Definition 5.22. Two definable sets are said to belong to the *same definable homeomorphism type* if there is a definable homeomorphism between the sets.

Example 5.23. In *Example 5.21*, we saw the sets in the family $(S_a)_{a\in\mathbb{R}^3}$ have four different definable homeomorphism types given by the empty set, a single point, two points, or all of $\mathbb{R}$. ♢

More generally, for o-minimal expansions of ordered fields, the sets of a definable family fall into only *finitely* many definable homeomorphism types.

To prove this result we will use the idea of *trivialization*:

Let $L = \{<, 0, 1, +, -, \times\}$, and let $\mathfrak{A}$ be an L-structure where $\mathfrak{A}$ is an expansion of an ordered (necessarily real closed, by *Theorem 4.19*) field.

Definition 5.24. Let $A \subseteq |\mathfrak{A}|^m$ and $S \subseteq |\mathfrak{A}|^n$ be definable sets and $f : S \to A$ be a definable map. A *definable trivialization of* f is a pair (F, λ), where $F \subseteq |\mathfrak{A}|^N$ is a definable set (for some N) and $\lambda : S \to F$ is a definable map such that $(f, \lambda) : S \to A \times F$ is a homeomorphism.

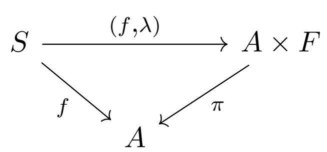
A will be refered to as the *base space.*

Definition 5.25. f is known as *definably trivial* if f has a definable trivialization. Moreover, given a definable subset A' of A, f is definably trivial *over* A' if

$$f|_{f^{-1}(A')} : f^{-1}(A') \to A'$$

is definably trivial.

Remark 5.26. If f is definably trivial, then the following diagram commutes:



so f "looks like" the projection map $\pi : A \times F \to A$. (Hence the name *trivialization*: f appears to be a trivial projection map.)

Moreover, (f, λ) maps each fiber $f^{-1}(a)$ of the family $(f^{-1}(a))_{a \in A}$ homeomorphically to $\{a\} \times F$, meaning each fiber of this family is definably homeomorphic to F. (This is an important point and will be used in *Theorem 5.29.*) ♢

Remark 5.27. If f is definably trivial, then it is definably trivial over any definable subset of its base space A. Given a definable subset $A' \subseteq A$, $(F, \lambda|_{f^{-1}(A')})$ is a definable trivialization of $f|_{f^{-1}(A')}$. ♢

Suppose we have a continuous definable map $f : S \to A$ between definable sets (as above). As one would expect, such maps in o-minimal structures are simple creatures as the following theorem shows:

Theorem 5.28. Trivialization theorem. *Consider f as above. We can partition A into definable subsets $A_1, \ldots, A_M$ such that f is definably trivial over each A_i.*

Proof.
See van den Dries [6, Chapter 9, Theorem 1.2]. ■

In essence, we can decompose A into parts such that f is essentially a trivial map over the parts.

We can now quickly prove what was hinted at in *Example 5.21:*

Corollary 5.29. *Let $S \subseteq |\mathfrak{A}|^{m+n}$ be a definable set with fibers S_a, where $a \in |\mathfrak{A}|^m$. The sets S_a belong to only finitely many distinct definable homeomorphism types.*

Proof.
Consider the projection map $\pi : |\mathfrak{A}|^{m+n} \to |\mathfrak{A}|^m$ and let $f = \pi|_S$. By *Theorem 5.28* we can partition $|\mathfrak{A}|^m$ into $A_1, \ldots, A_M$ such that f is definably trivial over each A_i. Say f has a definable trivialization (F_i, λ_i) over A_i. By *Remark 5.26*, for all $a \in A_i$, $S_a = f^{-1}(a)$ is definably homeomorphic to F_i. Since there are finitely many F_i, we conclude there are finitely many distinct definable homeomorphism types as required. ■

6. Topology in O-Minimal Structures

In this section, we explore the aspects of point set topology available to us in o-minimal structures. Here we will fix a language $L = \{<, 0, -, +\}$ and an o-minimal L-structure $\mathfrak{A}$ satisfying the axioms of an ordered group (which, as was proven in §4.3, is abelian and divisible). For more information and details, the author advises turning to Chapter 6 of *Tame Topology* [6] for aspects of o-minimality and Munkres [22] for aspects of topology.

For convenience, we will expand the language of the structure to include a constant 1 and a function $|\cdot|$ such that for all $x \in |\mathfrak{A}|$,

$$|x| := \begin{cases} x & \text{if } x \geq 0. \\ -x & \text{if } x < 0. \end{cases}$$

Remark 6.1. For our purposes, we do not need any of the properties that the absolute value function normally has (such as the triangle inequality, for example). As we are working in an *ordered* structure, we just want the $|\cdot|$ function to 'see' whether an element is ordered in front of 0 or not. Thus we are not defining a metric on this space, just a function whose output is positive. ♢

6.1. Definable Choice and Curve Selection

From a model theory perspective, we can first discuss *Skolem functions.*

Definition 6.2. Let L be a language and T an L-theory. T has *(built-in) Skolem functions* if for every formula $\phi(x, y)$ (with parameters $y \in |\mathfrak{A}|^n$) there is a function symbol f such that

$$T \vDash \forall y \ (\exists x \phi(x, y) \to \phi(f(y), y)). \tag{7}$$

Essentially we can 'remove' the existential statements in favour of statements including functions f. If L is not a large enough language to allow T to have built-in Skolem functions, we can expand L by adding in the necessary function symbols (and no longer say the Skolem functions are *built-in*). If this is the case, and the Skolem functions are definable, we say T has *definable Skolem functions.*

Example 6.3. In the theory of groups, we can *skolemize* the following sentences:

- $\exists x \forall y (x * y = y \wedge y * x = y)$ becomes $\forall y (1 * y = y \wedge y * 1 = y)$, where the constant function "1" has been introduced.
- $\forall x \exists y (x * y = 1 \wedge y * x = 1)$ becomes $\forall x (x * x^{-1} = 1 \wedge x^{-1} * x = 1)$, where the function $f(x) = x^{-1}$ has been introduced.

♢

We can prove in fact that our o-minimal structure $\mathfrak{A}$ has definable Skolem functions. In *Tame Topology* [6], the existence of such functions is known as *definable choice.*

Theorem 6.4. Definable Choice. *Let* $S \subseteq |\mathfrak{A}|^{m+n}$ *be definable, and let* $\Pi : |\mathfrak{A}|^{m+n} \to |\mathfrak{A}|^m$ *be the projection to the first* m *coordinates. There exists a definable map* $f : \Pi(S) \to |\mathfrak{A}|^n$ *such that* $\Gamma(f) \subseteq S$.

Proof.
This is a slightly misleading proof but aptly named. The main construction is to show we can definably choose an element in a definable set. We do so using the fact $\mathfrak{A}$ is equipped with a group structure as follows:
Let $X \subseteq |\mathfrak{A}|$ be definable and nonempty. We shall inductively define how we choose the element $e(X)$.

(1) If X has a least element, let $e(X)$ be this least element.

(2) If X does not have a least element, define the *left-most interval* (a, b) by:

$$a := \inf X, \qquad b := \sup_{x \in |\mathfrak{A}|} \{(a, x) \subseteq |\mathfrak{A}|\}$$

Choose $e(X)$ as

$$e(X) := \begin{cases} 0 & \text{if } a = -\infty,\, b = +\infty. \\ b - 1 & \text{if } a = \infty,\, b \in |\mathfrak{A}|. \\ a + 1 & \text{if } a \in |\mathfrak{A}|,\, b = +\infty. \\ \frac{a+b}{2} & \text{if } a, b \in |\mathfrak{A}|. \end{cases} \tag{8}$$

(3) Let $X \subseteq |\mathfrak{A}|^{m+1}$ be definable and nonempty with $m > 0$. Let π be usual projection map. By induction, we assume we can choose the element $a = e(\pi X)$. Then the fiber X_a is a subset of $|\mathfrak{A}|$, and we define $e(X) := (a, e(X_a))$.

Define $f(x) = e(S_x)$ for $x \in \Pi(S)$. Then for all x, $f(x) \in |\mathfrak{A}|^n$ and by definition of $e(S_x)$, we conclude $\Gamma(f) \subseteq S$ as required. ■

Remark 6.5. Skolem functions. Recall that projection in VDD structures is analogous to applying the existential quantifier in a model-theoretic L-structure. The existence of the map f in the statement of *Theorem 6.4* is comparable to (7). ♢

Remark 6.6. For $x \in |\mathfrak{A}|^m$ we can use the supnorm

$$|x| = |(x_1, \ldots, x_m)| := \max\{|x_1|, \ldots, |x_m|\}$$

when required. ♢

Corollary 6.7. Curve selection. *Let* X *be a definable set. If* $a \in \mathrm{cl}(X) \setminus X$ *then there is a definable continuous injective map* $\gamma : (0, \epsilon) \to X$ *("a curve") for some* $\epsilon > 0$ *such that* $\lim_{t \to 0} \gamma(t) = a$.

Proof.
Let $a \in \operatorname{cl}(X) \setminus X$. The definable set $\{|a - x| : x \in X\}$ contains arbitrarily small positive elements, so by o-minimality it contains an interval $(0, \epsilon)$ where $\epsilon > 0$. By definition for all $t \in (0, \epsilon)$ there exists $x \in X$ such that $|a - x| = t$. By definable choice/definable Skolem functions, there is then a definable map $\gamma : (0, \epsilon) \to X$ such that for all $t \in (0, \epsilon)$, $|a - \gamma(t)| = t$. By the *Monotonicity theorem*, we can assume γ is continuous (by decreasing ϵ if necessary). By construction γ is injective:

$$\gamma(t_1) = \gamma(t_2) \Rightarrow |a - \gamma(t_1)| = |a - \gamma(t_2)| \Rightarrow t_1 = t_2$$

and $\lim_{t \to 0} \gamma(t) = a$ as required. ■

From definable choice and curve selection, a bounty of topological properties follow presented by van den Dries [6, Chapter 6]:

Proposition 6.8. *Let C be a bounded cell in $|\mathfrak{A}|^m$, $m > 1$. Then $\pi \operatorname{cl}(C) = \operatorname{cl}(\pi C)$.* ■

Remark 6.9. When we speak about *continuous* functions, we have the definition in terms of limits available to us due to the norm:

$$f \text{ is continuous at } a \quad \Leftrightarrow \quad \forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall x \left(|x - a| < \delta \to |f(x) - f(a)| < \epsilon\right).$$

This notion of continuity coincides with continuity on the interval topology. ◊

Proposition 6.10. *If $f : X \to |\mathfrak{A}|^n$ is a continuous, definable map on a closed bounded set $X \subseteq |\mathfrak{A}|^m$, then the image $f(X)$ is closed and bounded in $|\mathfrak{A}|^n$.* ■

Corollary 6.11. "Extreme Value" theorem. *If $f : X \to |\mathfrak{A}|$ is a continuous, definable map on a (nonempty) closed bounded set $X \subseteq |\mathfrak{A}|^m$, then f assumes a maximum and a minimum value.* ■

Corollary 6.12. *Let $f : X \to |\mathfrak{A}|^n$ be a definable, continuous map on a closed bounded subset $X \subseteq |\mathfrak{A}|^m$. Then:*

(1) *A definable set $S \subseteq f(X)$ is closed $\Leftrightarrow$ $f^{-1}(S)$ is closed.*

(1) *A definable map $g : f(X) \to |\mathfrak{A}|^p$ is continuous $\Leftrightarrow$ $g \circ f : X \to |\mathfrak{A}|^p$ is continuous.*

■

6.2. Definable Paths, Partitions of Unity and Definable Curves

A variant on the idea of a definable curve is that of a *definable path*:

Definition 6.13. Let $X \subseteq |\mathfrak{A}|^m$. (Note that we are not requiring X to be definable.)

- A *definable path in* X is a definable continuous map $\gamma : [a,b] \to X$ with $a, b \in |\mathfrak{A}|$ and $a < b$.
- γ is said to *connect* the points $\gamma(a)$ and $\gamma(b)$.
- If $\gamma : [a,b] \to X$, $\delta : [b,c] \to X$ are definable paths in X with $\gamma(b) = \delta(b)$, then we can concatenate these paths and form $\gamma \vee \delta : [a,c] \to X$.
- X is *definably path connected* if every two points in the definable set X can be connected by a definable path.

Lemma 6.14. *If a definable set X is definably path-connected, then it is definably connected.*

Sketch Proof.
If X were definable and not connected, say $X = U \cup V$ with $U \cap V = \emptyset$, U, V open and nonempty, then choosing $u \in U$ and $v \in V$ we could not connect u and v by a definable path as there would be a 'gap' between U and V, which a continuous path map γ would not be able to bridge. ■

In general the converse (connected implies path connected) is not true. In dealing with the o-minimal expansion of an ordered abelian group, however, the converse is in fact provable. We will proceed via induction and use the *Cell Decomposition theorem.*

Lemma 6.15. *If the definable set X is definably connected, then it is definably path connected.*

Proof.
We can assume WLOG that X is an open cell in $|\mathfrak{A}|^m$. For $m = 1$, by o-minimality X is convex, and we are done. For $X \subseteq |\mathfrak{A}|^{m+1}$, let $\mathbb{X} = \pi(X) \subseteq |\mathfrak{A}|^m$ such that $X = (f,g)_{\mathbb{X}}$, where $f, g \in C(\mathbb{X})$. The case of f and/or g being $\pm\infty$ is not so different to what follows, where instead we would take our cues from (8). We will attempt to connect points (y,r) and (z,s) in X by a definable path.

See *fig. 5* overleaf.

We first note the 'vertical' paths

$$y \mapsto \frac{f(y)+g(y)}{2} \qquad \text{and} \qquad z \mapsto \frac{f(z)+g(z)}{2}.$$

Now the problem has become to join $\left(y, \frac{f(y)+g(y)}{2}\right)$ and $\left(z, \frac{f(z)+g(z)}{2}\right)$, but the path $\gamma : [a,b] \to \mathbb{X}$ joining y to z can be lifted to a path δ in X by

$$t \longmapsto \left(\gamma(t), \frac{f(\gamma(t)) + g(\gamma(t))}{2}\right).$$

Concatenating these three paths gives a definable path (y,r) to (z,s) in X, as required. ■

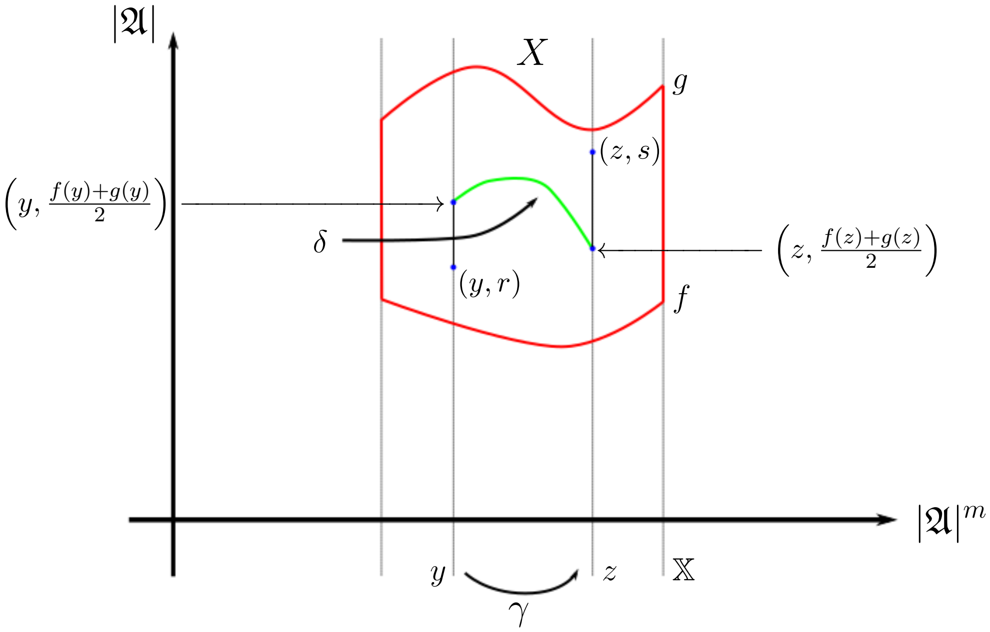


Figure 5

Continuing on we see that o-minimal structures (almost) admit *partitions of unity*:

Theorem 6.16. *Let $\{U_1, \dots, U_n\}$ be a finite collection of definable open sets covering a definable set $B \subseteq |\mathfrak{A}|^m$. There is a family of definable continuous functions $(f_i)_{i=1}^n$, $f_i : B \to [0,1]$ such that:*

(1) $\operatorname{supp} f_i \subseteq U_i$ *for all i, where the support of f_i*

$$\operatorname{supp} f_i := \operatorname{cl}(\{x \in B : f_i(x) \neq 0\}).$$

(2) $\forall x \in B$, $\sum_{i=1}^n f_i(x) > 0$.

The proof of this theorem follows immediately from the application of the following two lemmas [6, Chapter 6]:

Lemma 6.17. Shrinking of open coverings. *Suppose B and $\{U_i\}$ are as in* Theorem 6.16. *Then B is also a union of the definable open subsets $V_1, \dots, V_n$ with $\operatorname{cl}_B(V_i) \subseteq U_i$ for $1 \leq i \leq n$.*

Lemma 6.18. *Let $A \subseteq B \subseteq |\mathfrak{A}|^m$ be definable sets with A closed in B. Then there is a definable continuous function $f : B \to [0,1]$, where A is the kernel of f, i.e. $f^{-1}(0) = A$.*

Sketch proof of Theorem 6.16*:*
"Shrink" the covering $\{U_1, \dots, U_n\}$ of B to obtain another covering $\{V_1, \dots, V_n\}$ with

$\mathrm{cl}(V_i) \subseteq U_i$ for $i = 1, \ldots, n$. By *Lemma 6.18* there are definable functions $f_i : B \to [0,1]$ such that $f_i^{-1}(0) = B \setminus V_i$. Thus, $\operatorname{supp} f_i \subseteq \mathrm{cl}(V_i) \subseteq U_i$ for all i. Finally, as the collection $\{V_i\}$ covers B, $\sum_{i=1}^{n} f_i(x) > 0$ for all $x \in B$ as required. ■

Remark 6.19. If the functions in *Theorem 6.16* have the property

$$\forall x \in B, \ \sum_{i=1}^{n} f_i(x) = 1$$

instead of condition (2), the family of functions $(f_i)_{i=1}^{n}$ is said to be a *definable partition of unity for the covering* $U_1, \ldots, U_n$. In particular, if we expand $\mathfrak{A}$ to model RCF, then *Lemma 6.18* and *Theorem 6.16* can be generalised to prove that if B is covered by a finite collection of definable open sets $\{U_1, \ldots, U_n\}$, then there exists a definable partition of unity for the covering $\{U_1, \ldots, U_n\}$ [6, Chapter 6, Lemma 3.7]. ♢

In a metric space, a point in the closure of a set is the limit of a sequence of elements in that set. In $\mathfrak{A}$, we do not consider sequences but instead use an adequate substitution: *curves*.

Definition 6.20. Let X be a definable subset of $|\mathfrak{A}|^m$. A *definable curve in* X is a definable map $\gamma : I \to X$ for some interval $I = (a, b) \subseteq |\mathfrak{A}|$.

Our interest lies in the behaviour of γ at one of its endpoints, which by convention is the right endpoint $b \in |\mathfrak{A}| \cup \{+\infty\}$.

Remark 6.21. We do not require γ to be continuous. By the *Monotonicity theorem*, however, γ will be continuous on some subinterval (a', b) (specifically with the same right endpoint). ♢

Definition 6.22. Let $\gamma : I \to X$ be a definable curve in X. Given $p \in |\mathfrak{A}|^m$, we say $\gamma \to p$ if $\lim_{t \to b} \gamma(t) = p$. (It is not required for p to be an element of X.)
Moreover, we say γ is *completable* if there is a point p such that $\gamma \to p$. If $p \in X$ then γ is *completable in* X.

The following lemma consisting of easily proven results outlines the behaviour of definable curves:

Lemma 6.23.

- *If $\gamma \to p$, then p is unique, i.e. if $\gamma \to q$ then $p = q$.*
- *If X is bounded then γ is completable (by* Corollary 5.3*).*
- *If X is closed and bounded then γ is completable in X.*
- *On the subinterval (a', b) γ is either injective or constant (by the* Monotonicity theorem*).*

- *If $f : X \to Y$ is a definable map between definable sets, then $f \circ \gamma$ is a definable curve in Y.*
- *If the above f is surjective, then each definable curve δ in Y can be 'lifted' to a definable curve β in X. That is, for every definable curve δ in Y, there exists a definable curve β in X such that $f(\beta) = \delta$. (This point follows from* definable choice.*)* ■

As sequences in a metric space have their use in proving continuity, so do definable curves:

Lemma 6.24. *Let $f : X \to Y$ be a definable map between definable sets. Let $p \in X$. Then f is continuous at the point p if and only if for every definable curve $\gamma \to p$ in X, $f(\gamma) \to f(p)$ in Y.*

Proof.
The forward implication follows immediately from the behaviour of continuous functions with limits. We shall prove the reverse implication by proving the contrapositive. Suppose f is not continuous at p. Then $\exists \epsilon > 0$ such that the set

$$\{|x - p| : x \in X, |f(x) - f(p)| \geq \epsilon\}$$

contains arbitrary small finite elements. Therefore, by o-minimality there is an interval $(0, \delta) \subseteq \{|x - p| : x \in X, |f(x) - f(p)| \geq \epsilon\}$. As in the proof of *Corollary 6.7*, by definable choice/definable Skolem functions, there is a definable curve $\gamma : (0, \delta) \to X$ such that for all $t \in (0, \delta)$, $|\gamma(t) - p| = t$ and $|f(\gamma(t)) - f(p)| \geq \epsilon$. Then (reversing γ so t approaches a right endpoint) $\gamma \to p$ without $f(\gamma) \to f(p)$, as required for the contrapositive. ■

7. The Łojasiewicz Inequality

In this section, we will present two proofs of the Łojasiewicz inequality, the first using analytical techniques by Bierstone and Milman [2] and the second using o-minimality by van den Dries and Miller [8]. Not only will this 'compare and contrast' approach highlight the slickness that comes with o-minimality but the result, *Theorem 7.19*, will provide fodder for §8.

7.1. Semianalytic and Subanalytic sets

First, recall some definitions presented in §1:

Definition 7.1. A subset of $\mathbb{R}^n$ is *semialgebraic* if it is a Boolean combination of solution sets of polynomial equations $p(x_1, \ldots, x_n) = 0$ and polynomial inequalities $p(x_1, \ldots, x_n) > 0$.

Let $\mathcal{M}$ be a real analytic manifold.

Definition 7.2. A subset $X \subset \mathcal{M}$ is ***semi**analytic* if and only if for all $a \in \mathcal{M}$ there exists a neighbourhood U of a such that $X \cap U$ is a finite Boolean combination of solution sets of equations $p(x_1, \ldots, x_n) = 0$ and inequalities $p(x_1, \ldots, x_n) > 0$, where p is a real analytic function.

Definition 7.3. A subset $X \subset \mathcal{M}$ is ***sub**analytic* if and only if for all $a \in \mathcal{M}$ there exists a neighbourhood U of a such that $X \cap U$ is the projection of a relatively compact semianalytic set.
That is, there exists a real analytic manifold $\mathcal{N}$ and a relatively compact semianalytic subset Y of $\mathcal{M} \times \mathcal{N}$ such that

$$X \cap U = \pi(Y)$$

where $\pi : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ is the standard projection.

Subanalytic sets are a broader class of sets than semianalytic sets (in particular, every semianalytic set is trivially subanalytic).
The semianalytic sets have a fatal flaw: The Tarski-Seidenberg theorem is not true of these sets (that is, the projection of a semianalytic set may not be semianalytic).

Example 7.4. Consider the following example by Marker [19, §1]:

$$X = \left\{ \left(\frac{1}{n}, n\right) : n = 1, 2, 3, \ldots \right\}.$$

As a subset of $\mathbb{R}^2$, X is semianalytic by *Definition 7.2*, but its projection onto $\mathbb{R}$, $\pi(X) = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$, is not semianalytic at 0. $\diamondsuit$

Thus, we consider the subanalytic sets, which have nice properties carried over from semianalytic geometry but are also stable under projection. The subanalytic sets are a broader class of subsets of $\mathbb{R}^n$ than the semialgebraic sets, and we will see in what follows that they share a number of desirable properties with the semialgebraic sets. In §8 we will look for a broader collection of sets with the same desirable properties as the semialgebraic and subanalytic sets.

Section 1 of Bierstone and Milman [2] runs through many properties of semialgebraic sets, which should already be familiar to us from an o-minimal standpoint:

Theorem 7.5. *Let $P_1(x,y), \ldots, P_t(x,y)$ be polynomials, where $x = (x_1, \ldots, x_n)$. Then there is a semialgebraic partition $\{A_1, \ldots, A_m\}$ of $\mathbb{R}^n$ such that for each $k = 1, \ldots, m$, the zeros of $P_1, \ldots, P_t$ on A_k are given by continuous semialgebraic functions $\xi_1 < \cdots < \xi_{r_k}$ and the sign of each $P_j(x,y)$ on A_k depends only on the signs of $y - \xi_i(x)$, $i = 1, \ldots, r_k$.*

Proof.
See Bierstone and Milman [2, Corollary 1.4]. It should be noted that this result is a sheep in a wolf's clothing: this theorem motivates the notion of 'cell decomposition' in the *Cell Decomposition theorem* (*Theorem 5.11*), and the functions ξ_i correspond to $\Gamma(f_i)$ in the cell decomposition. ■

From this theorem, we immediately obtain the *Tarski-Seidenberg theorem*[6]:

Corollary 7.6. Tarski-Seidenberg theorem. *The image of a semialgebraic set $X \subseteq \mathbb{R}^{n+1}$ by the projection map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ is semialgebraic.*

Proof.
As X is semialgebraic it is of the form $X = \cup_{i=1}^p \cap_{j=1}^q \{P_{ij}(x,y)\ \sigma_{ij}\ 0\}$ where each P_{ij} is a polynomial and σ_{ij} is the common notation to denote either $>$ or $=$. Applying *Theorem 7.5* to the boundary of X (the polynomials P_{ij}) the projection of X will be a union of the relevant A_k, as required. ■

A second basic result forming the foundations of semialgebraic geometry is *Thom's lemma*:

Lemma 7.7. Thom's lemma. *Let $P_1(x), \ldots, P_m(x)$ be a finite family of polynomials in one variable stable under differentiation. Let*

$$A = \bigcap_{i=1}^{m} \{x \in \mathbb{R} : P_i(x)\ \sigma_i\ 0\},$$

where each σ_i is either $>$, $<$, or $=$. Then:

(1) *A is either empty or connected.*

[6]Proven this time without reference to quantifier elimination or o-minimality.

(2) *If A is nonempty then*

$$\mathrm{cl}(A) = \bigcap_{i=1}^{m} \{x \in U : P_i(x)\ \bar{\sigma_i}\ 0\},$$

where $\bar{\sigma_i}$ is $\geq$, $\leq$ or $=$ as σ_i is $>$, $<$ or $=$ respectively.

Proof.
See Bierstone and Milman [2, Lemma 1.9]. ∎

Thom's lemma makes a reappearance in *semianalytic* geometry, this time under the guise of *separating families*:

Definition 7.8. Let U be an open subset of $\mathcal{M}$. A finite family $f_1, \ldots, f_m$ of real analytic functions on U is *separating* if for any semianalytic subset $A \subseteq U$ of the form

$$A = \bigcap_{i=1}^{m} \{x \in U : f_i(x)\ \sigma_i\ 0\},$$

where each σ_i is either $>$, $<$ or $=$, we have:

(1) A is either empty or connected.

(2) If A is nonempty then the closure of A in U is given by

$$\mathrm{cl}(A) = \bigcap_{i=1}^{m} \{x \in U : f_i(x)\ \bar{\sigma_i}\ 0\},$$

where $\bar{\sigma_i}$ is $\geq$, $\leq$ or $=$ as σ_i is $>$, $<$ or $=$ respectively.

Theorem 7.9. *Any finite family of analytic functions on a real analytic manifold $\mathcal{M}$ can be completed, in some neighbourhood of a given point, to a separating family.*

Proof.
See Bierstone and Milman [2, Theorem 2.6]. ∎

Remark 7.10. Bierstone and Milman [2, §3] continue describing the nice properties of subanalytic sets, many of which are identical to the semianalytic case:

(1) Every connected component of a subanalytic set is subanalytic.

(2) A family of sets is said to be *locally finite* (in a topological space X) if for all $x \in X$ there exists a neighbourhood U such that U intersects only finitely many members of the family.
As it turns out, the family of connected components of a subanalytic set is locally finite.

(3) Every subanalytic set is locally connected.

(4) The closure, the complement, and thus the interior of a subanalytic set is subanalytic. In particular, the theorem of the complement ([2, Theorem 3.10]) is hard-won.

(5) The intersection and union of a finite collection of subanalytic sets is subanalytic (following from the semianalytic case from *Definition 7.2*).

Items (1)-(4) are all corollaries to *Theorem 7.9.* The 'niceness' of the geometry of semialgebraic and subanalytic sets will make an appearance again in §8. ♢

7.2. The Łojasiewicz Inequality

Another property indicating the subanalytic sets are well behaved is the *Łojasiewicz inequality*:

Theorem 7.11. Łojasiewicz Inequality. *Let $\mathcal{M}$ be a real analytic manifold and let $K \subseteq \mathcal{M}$. Let $f, g : K \to \mathbb{R}$ be subanalytic functions*[7] *with compact graphs. If $f^{-1}(0) \subseteq g^{-1}(0)$ then there exist $c, r > 0$ such that for all $x \in K$,*

$$|f(x)| \geq c|g(x)|^r.$$

In order to prove this theorem we will need to cite the following results:

Theorem 7.12. [2, Theorem 6.1] *Let X be a subanalytic subset of $\mathcal{M}$. Then:*

(1) *If $\dim X \leq 1$, X is semianalytic.*

(2) *If $\dim \mathcal{M} \leq 2$, X is semianalytic.* ■

Theorem 7.13. [2, Theorem 0.1] **Uniformization theorem**. *Let X be a subanalytic subset of $\mathcal{M}$ and suppose X is closed. Then there is a real analytic manifold $\mathcal{N}$ (of the same dimension as X) and a (proper) real analytic mapping $\varphi : \mathcal{N} \to \mathcal{M}$ such that $\varphi(\mathcal{N}) = X$.* ■

By *proper* we mean the preimage of every compact subset in $\mathcal{M}$ is compact in $\mathcal{N}$.

This large, complicated theorem has a more practical application: what was known as curve selection in o-minimal structures.

Definition 7.14. A topological space X is *locally connected at the point x* if every open set U containing x contains a connected open subset V with $x \in V \subseteq U$.

Corollary 7.15. "Curve Selection." *Let $A \subset \mathcal{M}$ be a one dimensional semianalytic set. Let $a \in \mathrm{cl}(A)$ and assume $A \setminus \{a\}$ is locally connected at a. Then $\exists \epsilon > 0$ and a real analytic mapping $\gamma : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\gamma(0) = a$ and $\gamma((0, \epsilon))$ is a neighbourhood of a in $A \setminus \{a\}$.* ■

[7] Functions whose graphs are subanalytic.

Proof.
See Bierstone and Milman [2, Lemma 6.3]. ∎

We now have the arsenal to prove the *Lojasiewicz inequality.*

Theorem 7.11. Łojasiewicz Inequality. *Let $\mathcal{M}$ be a real analytic manifold and let $K \subseteq \mathcal{M}$. Let $f, g : K \to \mathbb{R}$ be subanalytic functions with compact graphs. If $f^{-1}(0) \subseteq g^{-1}(0)$ then there exist $c, r > 0$ such that for all $x \in K$,*

$$|f(x)| \geq c|g(x)|^r.$$

Proof.
Define

$$L := \{(u, v) \in \mathbb{R}^2 : u = |g(x)|, v = |f(x)| \text{ for some } x \in K\}.$$

As f and g have compact graphs, L is compact. Furthermore, L is semianalytic by *Theorem 7.12.* Let $\pi(u, v) = u$ be the standard projection and assume $0 \in \pi(L)$ is not an isolated point of $\pi(L)$.

By *Corollary 7.15*, there exists $\epsilon > 0$ and a parametrised analytic curve

$$\gamma : (-2\epsilon, 2\epsilon) \to L, \qquad \gamma(s) = (u(s), v(s))$$

such that $u(0) = 0$, if $s > 0$ then $u(s) > 0$ (by making ϵ appropriately small), and

$$L \cap ([0, u(\epsilon)) \times \mathbb{R}) \text{ is bounded below by } \gamma([0, \epsilon)) \qquad (\text{see } \textit{fig. 6}). \tag{9}$$

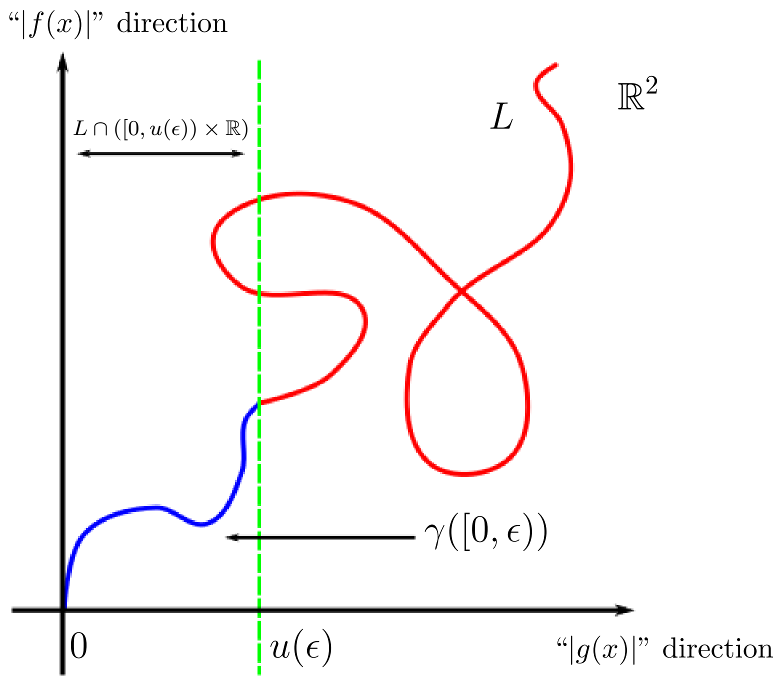


Figure 6

By a change of parameter s, we can assume $u(s) = s^k$ for some positive integer k. In other words in some neighbourhood u is a convergent power series, which we can write as $u(s) = s^k \times unit$, where under an appropriate change of parameter we can forget about the unit. Also $v(s)$ is strictly positive on $(0, \epsilon)$ as for all $u \in (0, \epsilon^k)$ the set

$$\{x \in K : |g(x)| = u\}$$

is a compact set on which $|f(x)|$ does not vanish (due to the condition $f^{-1}(0) \subseteq g^{-1}(0)$), thus $v = |f(x)|$ has a nonzero minimum.

Let $\delta = \epsilon^k$. Then by condition (9):

$$v = |f(x)| \geq v(|g(x)|^{\frac{1}{k}}) = v(s) > 0 \qquad \text{whenever } 0 < |g(x)| = u < \delta,$$

and therefore as v is analytic, there exist $c, r > 0$ such that $|f(x)| \geq c|g(x)|^r$ whenever $|g(x)| \leq \frac{\delta}{2}$.

Finally, $\{x \in K : |g(x)| \geq \frac{\delta}{2}\}$ is a compact set on which $|f(x)|$ does not vanish (again as $f^{-1}(0) \subseteq g^{-1}(0)$), so the inequality can be expanded to all of K, perhaps after reducing c. ■

Remark 7.16. On that last point, "the inequality can be expanded to all of K, perhaps after reducing c". Informally we can argue as follows:

If $|f(x)|$ at some x is less than $c|g(x)|^r$ we can reduce c such that the inequality, $|f(x)| \geq c'|g(x)|^r$, is still true. If $|f(x)|$ is less than $c|g(x)|^r$ and it cannot be rescued by decreasing c, then a condition in the statement of the theorem has been contradicted: f and/or g is not subanalytic or $f^{-1}(0) \nsubseteq g^{-1}(0)$. ◊

Remark 7.17. In particular, setting $\mathcal{M} = \mathbb{R}^n$, $X = f^{-1}(0)$ and

$$g(x) = d(x, X) := \min_{z \in \mathrm{cl}(X)} |x - z|,$$

we obtain for all $x \in \mathbb{R}^n$,

$$|f(x)| \geq c \cdot d(x, X)^r.$$

(Note $d(x, X)$ is subanalytic by [8, Remark 3.11].) ◊

7.3. The O-Minimal Approach

The difficulty in proving the Łojasiewicz inequality classically should not be understated: *Theorems 7.12 & 7.13* are hard-won and rely on many other ideas and lemmas. It seems cruel that such heavy machinery and powerful results are needed to prove such a small inequality; the phrase "swatting a fly with a sledgehammer" is appropriate in this context. Another approach to prove this result is taken via the o-minimality route [8].

First, some notation:

Remark 7.18. Define:

$$\Phi^p_{\mathfrak{A}} := \left\{ \phi : \mathbb{R} \to \mathbb{R} : \quad \phi \text{ is } \begin{array}{l} \cdot \text{ an odd, strictly increasing bijection,} \\ \cdot \text{ definable in } \mathfrak{A}, \\ \cdot\ C^p \text{ on } \mathbb{R} \text{ and } p\text{-flat at } 0. \end{array} \right\}$$

Where "p-flat at 0" means

$$\frac{d^k}{dx^k}\phi(0) = \phi^{(k)}(0) = 0 \qquad \text{for } k = 0, \ldots, p.$$

$\Diamond$

Consider an o-minimal expansion of the ordered field of real numbers. We shall prove the *Generalised Łojasiewicz inequality*:

Theorem 7.19. Generalised Łojasiewicz inequality. *Let $f, g : A \to \mathbb{R}$ be definable continuous functions with $f^{-1}(0) \subseteq g^{-1}(0)$ and $A \subseteq \mathbb{R}^n$ compact. Then there exists $\phi \in \Phi^p_{\mathfrak{A}}$ such that for all $x \in A$,*

$$|\phi(g(x))| \le |f(x)|.$$

Proof.
See van den Dries & Miller [8, Appendix C] for the full details.
The trickiest part of this theorem is the proper choice of $\phi \in \Phi^p_{\mathfrak{A}}$. We can do so as follows:

Fix $y \in g^{-1}(0)$ and define $U := A \setminus g^{-1}(0)$. Furthermore, define

$$A(y,t) := \{x \in A : |x - y| \le 1 \text{ and } |g(x)| = t\}.$$

(see *fig. 7* for an example of what this may look like).

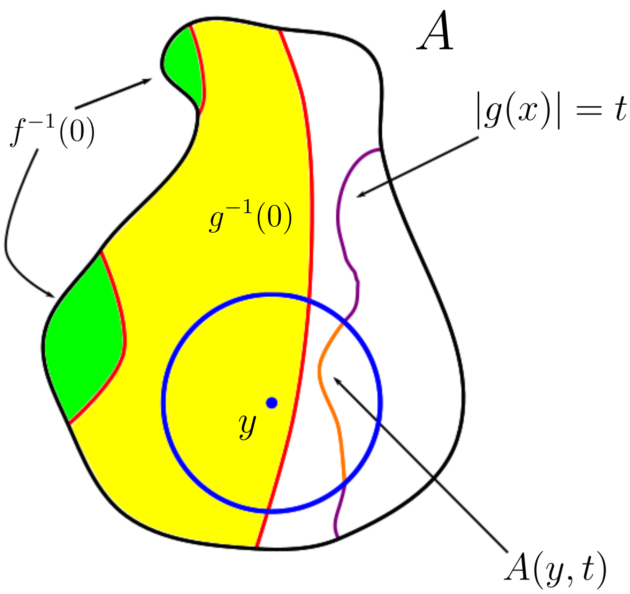


Figure 7

Restrict f to U. Then define F by:

$$F(y,t) := \begin{cases} \max\left\{\frac{1}{|f(x)|} : x \in A(y,t)\right\} & \text{if } A(y,t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

We wish to restrict the behaviour of f near $g^{-1}(0)$, i.e. when t is small. To that end, we want to find $\varphi \in \Phi^p_{\mathfrak{A}}$ such that $\lim_{t\to 0^+} \varphi(t)F(y,t) = 0$.

We can break the argument into two cases:

Case 1. If $\lim_{t\to 0^+} F(y,t) = +\infty$, define

$$H(t) := \left|\frac{1}{F(y,t)}\right|.$$

We break this case into two further subcases:

Subcase 1 If there exists $n \in \mathbb{N}$ such that $H(t) \geq t^n$ as $t \to 0^+$ we take $\theta(t) = t^m$, where $m \in \mathbb{N}$ is an odd integer strictly greater than n. Then $\theta \in \Phi^p_{\mathfrak{A}}$ for some p and $\theta(t) < H(t)$ for all sufficiently small t.

Subcase 2 Suppose for all $n \in \mathbb{N}$, $\lim_{t\to 0^+} t^{-n}H(t) = 0$. By cell decomposition, we can assume H is C^p on $(0,a)$ and $0 < a < 1$. Consider

$$\theta(t) = tH\left(\frac{at^2}{(1+t^2)}\right).$$

θ is still C^p on $\mathbb{R}\setminus\{0\}$, $\theta(t) < H(t)$ for sufficiently small t, and we still have $\lim_{t\to 0^+} t^{-k}\theta(t) = 0$ for $k = 0,\dots,p$. Therefore, by L'Hôspital's rule, we have $\lim_{t\to 0^+} \theta^{(k)}(t) = 0$. Hence $\theta \in \Phi^p_{\mathfrak{A}}$ as required.

Note that in either case we obtain $\theta(t) < H(t)$. Using previous notation, this inequality is equivalent to

$$\theta(t) < \frac{1}{|F(t)|} \qquad (\Leftrightarrow \qquad \theta(t)|F(t)| < 1\ ) \qquad \text{for } t \text{ small},$$

thus setting $\varphi = \theta^3$ gives us $\lim_{t\to 0^+} \varphi(t)F(y,t) = 0$, as required.

Case 2. On the other hand, if $F(y,t)$ is bounded as $t \to 0^+$, then the fact that we can choose θ such that

$$\theta(t)\,|F(y,t)| < 1 \qquad \text{for } t \text{ small}$$

is clear. We obtain again $\lim_{t\to 0^+} \varphi(t)F(y,t) = 0$, as required.

With these cases in mind,

$$\begin{aligned}
&\lim_{t\to 0^+} \varphi(t)F(y,t) = 0 \\
\Leftrightarrow \quad & \lim_{|g(x)|\to 0} \varphi(|g(x)|) \max\left\{\tfrac{1}{|f(x)|}\right\} = 0 \quad \text{for } x \in A(y, |g(x)|) \\
\Leftrightarrow \quad & \lim_{x\to y} |\varphi(g(x))| \max\left\{\tfrac{1}{|f(x)|}\right\} = 0 \\
\Leftrightarrow \quad & \lim_{x\to y} \frac{\varphi(g(x))}{f(x)} = 0. \qquad (10)
\end{aligned}$$

Now that we have an appropriate $\varphi \in \Phi^p_{\mathfrak{A}}$, the remainder of the proof is nearly trivial. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x) := \begin{cases} \frac{\varphi(g(x))}{f(x)} & x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi \circ g = hf$. Set

$$C = 1 + \max\{|h(x)| : x \in A\} \qquad \text{and} \qquad \phi = \frac{\varphi}{C}.$$

C is finite as A is compact, and h is continuous due to (10) for $y \in \mathrm{bd}(g^{-1}(0))$, the boundary of $g^{-1}(0)$. Rewriting, we obtain for all $x \in A$,

$$|\phi(g(x))| \le |f(x)|$$

as required. ■

Example 7.20. Consider $f(x) = e^{-\frac{1}{x^2}}$ (where $f(0) = 0$), $g(x) = x$ on $A = [0, 2] \subset \mathbb{R}$. As $f^{-1}(0) = g^{-1}(0) = \{0\}$ we conclude $F(0,t) = e^{\frac{1}{t^2}}$. We are in **Subcase 1** of **Case 1**, meaning

$$\theta(t) = te^{-\frac{(1+t^2)^2}{a^2t^4}}$$

for $0 < a < 1$. This calculation indeed gives us $\theta \in \Phi^p_{\mathfrak{A}}$, leading to $\phi \in \Phi^p_{\mathfrak{A}}$ such that for all $x \in A$,

$$|\phi(g(x))| = |\phi(x)| \le |f(x)|$$

as required. ♢

Remark 7.21. Note that the *Generalised Łojasiewicz inequality* is a more flexible Łojasiewicz inequality than the one appearing in *Theorem 7.11*. The Łojasiewicz inequality of *Theorem 7.11* required f and g to be *subanalytic*, which $f(x) = e^{-\frac{1}{x^2}}$ is not. If we want a Łojasiewicz inequality such as

$$c|g(x)|^r \le |f(x)|$$

in o-minimal structures we need an addition condition that, although not very strict, is necessary. ♢

Definition 7.22. We call $\mathfrak{A}$ *polynomially bounded* if for every definable function f, there exists $N \in \mathbb{N}$ such that for all sufficiently large, positive x, $|f(x)| \leq x^N$.

In a very short paper [21], Miller proves two striking results:

Theorem 7.23. *Let $\mathfrak{A}$ be o-minimal and not polynomially bounded. Then the exponential function* exp *is definable.* ■

This theorem implies definable functions on $\mathfrak{A}$ fall into only two classes:

Corollary 7.24. *$\mathfrak{A}$ is either polynomially bounded or contains* exp*. If $\mathfrak{A}$ is polynomially bounded, then there exist $c, r \in \mathbb{R}$ with $c \neq 0$ such that $x \mapsto x^r$ is definable and $f(x) \to cx^r$ as $x \to +\infty$.* ■

To prove the 'classical' Łojasiewicz inequality of *Theorem 7.11* we will assume our structure $\mathfrak{A}$ is polynomially bounded. This assumption is again crucial as not every o-minimal structure over $\mathbb{R}$ is necessarily polynomially bounded (for example, due to Wilkie [27] $(\mathbb{R}, <, 0, 1, +, \exp)$ is o-minimal and contains exp).

Theorem 7.25. Łojasiewicz inequality. *Let $A \subseteq |\mathfrak{A}|^n$ be a definable compact set and let $f, g : A \to \mathbb{R}$ be definable and continuous with $f^{-1}(0) \subseteq g^{-1}(0)$. There exists $c, r > 0$ such that*

$$c|g(x)|^r \leq |f(x)| \qquad \textit{for all } x \in A.$$

Sketch proof.
This theorem follows from the proof of *Theorem 7.19:* as the functions are polynomially bounded we cannot be in **Subcase 2** of **Case 1**. We can see this heuristically as

$$H(t) \propto \frac{1}{F(y,t)} \propto f(x) \qquad \text{for } y \in g^{-1}(0),\, x \in A(y,t).$$

Suppose we are in **Subcase 2** of **Case 1**, that is for all $n \in \mathbb{N}$,

$$\lim_{t \to 0^+} t^{-n} H(t) \quad \left(\propto \quad \lim_{x \to 0^+} x^{-n} f(x) \right) \quad = 0.$$

Note that if $f(x)$ is definable then so is

$$g(x) := f\left(\frac{1}{x}\right)$$

meaning $g(x)$ is polynomially bounded as well. This means f has polynomial behaviour near 0, by *Corollary 7.24*. Therefore it is not possible for

$$\lim_{x \to 0^+} x^{-n} f(x) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

The θ we acquire from **Case 1, Subcase 1** or **Case 2** is of the form $\theta(x) = x^m$. Finishing the proof of *Theorem 7.19* with $c = \frac{1}{C}$ we conclude (for some r related to m)

$$c|g(x)|^r \leq |f(x)| \qquad \forall x \in A,$$

as required. ■

Corollary 7.26. As is suggested in *Lemma 3.27*, the function $d(x,X)$ from *Remark 7.17* is definable, meaning there exist $r, c > 0$ such that for all $x \in A$,

$$|f(x)| \geq c \cdot d(x,X)^r.$$

■

While there was some technical difficulty in obtaining $\phi \in \Phi^p_{\mathfrak{A}}$ for the *Generalized Łojasiewicz inequality*, we can see it is an easier proof requiring lighter results than the 'classical' *Łojasiewicz inequality*, *Theorem 7.11*.

Remark 7.27. Although we work with o-minimal structures, we are not as removed from subanalytic geometry as one might expect. Consider the following structure: $\mathfrak{R}_{\text{an}} = (\mathbb{R}, 0, 1, <, +, -, \times, (f_i)_{i\in I})$, where f_i ranges over all *restricted analytic functions*. That is, all functions

$$f : \mathbb{R}^n \to \mathbb{R}$$

(for all $n \in \mathbb{N}$) that vanish identically off $[-1,1]^n$ and whose restriction to $[-1,1]^n$ is analytic. The definable sets of $\mathfrak{R}_{\text{an}}$ are known as the *finitely subanalytic sets*. This structure is in fact an o-minimal structure [8]. ♢

A direct consequence of the *Generalized Łojasiewicz inequality* is *Generalized Hölder continuity*:

Corollary 7.28. Generalized Hölder continuity. *Let $f : A \to \mathbb{R}$ be a definable continuous function, where $A \subseteq \mathbb{R}^n$ is nonempty and compact. Then there exists $\phi \in \Phi^p_{\mathfrak{A}}$ such that*

$$|f(x) - f(y)| \leq \phi^{-1}(|x-y|)$$

for all $x, y \in A$.

Proof.
Consider the functions

$$\begin{aligned} |\cdot| : A^2 &\to \mathbb{R} & |f| : A^2 &\to \mathbb{R} \\ (x,y) &\mapsto |x-y| & (x,y) &\mapsto |f(x)-f(y)| \end{aligned}$$

Applying *Theorem 7.19* to these functions, we obtain $\theta \in \Phi^p_{\mathfrak{A}}$ and $C > 0$ such that for all $x, y \in A$,

$$\theta(|f(x)-f(y)|) \leq C|x-y|.$$

Setting $\phi = \frac{\theta}{C}$ we conclude

$$|f(x) - f(y)| \leq \phi^{-1}(|x-y|)$$

for all $x, y \in A$, as required. ■

Remark 7.29. In the same fashion as the Generalized Łojasiewicz inequality reduces to the Łojasiewicz inequality, so too does Generalized Hölder continuity reduce to Hölder continuity: that is, a function is *Hölder continuous* if there exist $C, r > 0$ such that for all $x, y \in A$,

$$|f(x) - f(y)| \leq C|x - y|^r.$$

◊

8. Collecting the Tame Properties of O-Minimal Structures

Grothendieck first presented the idea of *tame topology* in 1984 in his famous paper *Equisse d'un Programme* [16, pp. 5-48][8]. In his proposal for long term mathematical research, Grothendieck called for the recasting of topology to reflect the 'moderate' or 'tame' topological properties of the semialgebraic sets and thus avoid unpleasant results, like the Banach-Tarski Paradox, or unpleasant objects, like the topologist's sine curve (*Remark 5.4*) or space-filling curves [24]. A'Campo, Ji, and Papadopoulos [1] write:

> Grothendieck recalls that the field of topology at the time he wrote his Esquisse was still dominated by the development, done during the 1930s and 1940s, by analysts, in a way that fits their needs, rather than by geometers. He writes that the problem with such a development is that one has to deal with several pathological situations that have nothing to do with geometry. He declares that the fact that "the foundations of topology are inadequate is manifest from the very beginning, in the form of 'false problems' (at least from the point of view of the topological intuition of shape)."

There is no strict definition to what constitutes calling a property *tame*. Rather as this idea arose from observing the nice topological and geometric properties of the semialgebraic and semianalytic/subanalytic sets, we observe what properties common to these sets constitute preferable and advantageous behaviour, so that we might create a new field of topology devoted to the study of these nice objects. Moving to more advanced topological properties than the standard 'the intersection, closure, and complement of a "$\cdots$" set is a "$\cdots$" set', Marker [19] gives examples of tame properties such as:

(1) **Stratification**. If X is semialgebraic, then X can be written as a disjoint union

$$X = X_1 \cup \cdots \cup X_n$$

of semialgebraic sets, where each X_i is a connected real analytic manifold, and if $\text{cl}(X_i) \cap X_j \neq \emptyset$ for $i \neq j$, then $\text{cl}(X_i) \supseteq X_j$ and $\dim X_i > \dim X_j$.

(2) In particular, every semialgebraic set has finitely many connected components and the boundary of a semialgebraic set is a semialgebraic set of lower dimension.

(3) **Smooth maps**. If $f : X \to \mathbb{R}$ is semialgebraic, then X can be partitioned into finitely many disjoint semialgebraic sets X_i such that $f|_{X_i}$ is analytic.

(4) **Finiteness**. A semialgebraic family represents only finitely many semialgebraic homeomorphism types.

[8] Submitted in 1984, it was not published formally until 1997. Schneps and Lochak also provide an English translation [16, pp. 243-283].

More examples of tame properties have been noted by various authors [1, 8, 15, 16, 19, 29].

In the words of Grothendieck [16] presented at the start of this thesis:

> My approach toward possible foundations for a tame topology has been an axiomatic one. Rather than declaring (which would indeed be a perfectly sensible thing to do) that the desired "tame spaces" are no other than (say) . . . semianalytic spaces, and then developing in this context the toolbox of constructions and notions which are familiar from topology, supplemented with those which had not been developed up to now, for that very reason, I preferred to work on extracting which exactly, among the geometrical properties of the semianalytic sets in a space $\mathbb{R}^n$, make it possible to use these as local "models" for a notion of "tame space" (here semianalytic), and what (hopefully!) makes this notion flexible enough to use it effectively as the fundamental notion for a "tame topology" which would express with ease the topological intuition of shapes. Thus, once this necessary foundational work has been completed, there will appear not one "tame theory", but a vast infinity . . .

If *Esquisse* can therefore be seen as a challenge to describe a general class of sets that shares the tame topological properties of the semialgebraic and subanalytic sets, then according to Marker [19], "o-minimality is the model theoretic response to this challenge".

Compiling our results from previous sections, we see a sufficiently powerful[9] o-minimal L-structure has the following properties:

(1) **Stratification**. By the *Cell Decomposition theorem*, $(\mathrm{I})_{\mathrm{m}}$ every definable set can be partitioned into a union of cells C_i. Moreover, each cell is connected by *Theorem 5.15.* Another result of cell decomposition is if for some $i \neq j$, we have

$$\mathrm{cl}(C_i) \cap C_j \neq \emptyset$$

this implies

$$\mathrm{cl}(C_i) \supseteq C_j \text{ and } \dim C_i > \dim C_j.$$

(Dimension, although not discussed here, is discussed further in [6, Chapter 4].)

(2) Every definable set has finitely many connected components by *Corollary 5.17.* The boundary of a definable set is definable, and thus it is a definable set of a lower dimension as well.

(3) **Smooth maps**. By the *Cell Decomposition theorem*, $(\mathrm{II})_{\mathrm{m}}$ every definable function $f : A \to |\mathfrak{A}|$ has a cell decomposition C_i of A such that $f|_{C_i}$ is continuous.

Furthermore, for specific o-minimal structures on $(\mathbb{R}, +, \times)$, we can extend the cell decomposition presented in §5.2 to a decomposition into cells C_i such that $f|_{C_i}$ is C^p, for p a positive integer [8].

[9] As expanded upon in earlier sections, for these results we occasionally need the o-minimal structure to be an expansion of the theory of ordered groups, real closed fields, etc.

(4) In particular if (3) holds with $p = \omega$, $f|_{C_i}$ is analytic.

For example take the o-minimal structure $\mathfrak{R}_{\text{an,exp}} = (\mathfrak{R}_{\text{an}}, \exp)$. ($\mathfrak{R}_{\text{an}}$ is defined in *Remark 7.27.*) This structure admits an *analytic cell decomposition* [7], which is to say the cells in the decomposition are *analytic* (meaning the functions f, g in *Definition 5.5* are definable, continuous, and analytic). The *Cell Decomposition theorem*, $(\text{I})_\text{m}$ & $(\text{II})_\text{m}$ can be generalised in such a structure to include analytic maps [7, Theorem 8.8].

(5) **Finiteness**. In §5.3 we saw how the fibers of a definable family fall into finitely many definable homeomorphism types by *Corollary 5.29*.

As well as these semialgebraic and semianalytic/subanalytic properties, the definable sets in o-minimal structures also have a host of other 'tame' properties:

- By the *Monotonicity theorem*, all definable functions on intervals are constant or strictly monotonic and continuous (up to a finite number of points) and by *Corollary 5.3* the limits of definable functions exist (in $|\mathfrak{A}| \cup \{-\infty, +\infty\}$).
- We can extend this result for o-minimal structures on $(\mathbb{R}, +, \times)$ such that the restriction of f to each subinterval is C^p and not just continuous. As noted by van den Dries and Miller [8, §4], "for every presently-known o-minimal structure on $(\mathbb{R}, +, \times)$, [this result] holds with 'analytic' in place of 'C^p'".
- By the *Cell Decomposition theorem*, $(\text{III})_\text{m}$, there is a uniform bound on the size of fibers of definable sets.
- O-minimality is preserved under elementary equivalence by *Theorem 5.13* (and although this is not a topological property, it certainly is a good property to have).
- By trivialization (*Theorem 5.28*), similar to cell decomposition, given a definable map $f : S \to A$ between definable sets we can partition A into definable subsets such that f is 'trivial looking' over each subset of A (for o-minimal expansions of ordered fields).
- O-minimal structures have definable Skolem functions. This property is commonly refered to as *definable choice* (§6.1).
- Definable choice and curve selection (§6.1 and §6.2) allow us to use curves in place of sequences, leading to many standard topological properties (*Propositions 6.8 & 6.10, Corollary 6.12, Theorem 6.16, Lemmas 6.23 & 6.24*).
- In the o-minimal expansion of an ordered abelian group, a set is definably connected if and only if it is definably path connected (*Lemma 6.15*).
- If $\mathfrak{A} \models \text{RCF}$ then $\mathfrak{A}$ allows definable partitions of unity by *Remark 6.19.*

- *Remark 7.10* made about subanalytic sets is true for the definable sets of an o-minimal structure: the intersection, closure, complement, connected component, etc. of a definable set is definable.
- The *Generalized Łojasiewicz inequality* and the *Łojasiewicz inequality* (*Theorems 7.19 & 7.25*) hold in certain o-minimal structures as does *Generalized Hölder continuity* and *Hölder continuity* (*Theorem 7.28 and Remark 7.29*).
- The 'growth dichotomy' of definable functions (*Corollary 7.24*) places an impressive restriction on the behaviour of functions in o-minimal structures: they must be either polynomially bounded or exponential.

Much much more can be said on the tame features of o-minimal structures. The reader is invited to explore [8, §4 Appendix C] and [1, 15, 16, 19, 29] for another selection of properties.

We conclude there is ample evidence that the theory of o-minimal structures is a well grounded response to Grothendieck's call for *topologie modérée* and an area that deserves more attention going forward.

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