Homological Algebra: Presentation Notes

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1. Recalling From Previous Talks

Category definition. This is a very general definition for a structure; so much so that they were called "abstract nonsense" by Norman Steenrod. Aluffi retorts "[abstract nonsense] is essentially accurate and not necessarily derogatory: categories refer to nonsense in the sense that they are all about the 'structure', and not about the 'meaning', of what they represent." I'm forced to agree with both viewpoints: by defining a very broad structure, we can encompass many algebraic objects and instead focus on how they relate to other algebraic objects, however for the notion of category to be useful additional properties need to be introduced. *Abelian categories* have these additional properties and are commonly used as a basic object in this branch of mathematics.

Definition 1.1. A category $\mathfrak{C}$ is *abelian* if $\mathfrak{C}$ is additive and if each morphism $f \in \operatorname{Hom} \mathfrak{C}$ admits a kernel u and cokernel p and the thus induced morphism $\bar{f} : \operatorname{Coker} u \to \operatorname{Ker} p$ is an isomorphism.

Note that if every morphism in a category $\mathfrak{C}$ admits a kernel and cokernel, $\bar{f}$ naturally exists, is unique, and makes the following diagram commute:

$$\begin{array}{ccccccc} \operatorname{Ker} f & \xrightarrow{u} & X & \xrightarrow{f} & Y & \xrightarrow{p} & \operatorname{Coker} f \\ & & {\scriptstyle p'}\downarrow & & \uparrow{\scriptstyle u'} & & \\ & & \operatorname{Coker} u & \xrightarrow{\bar{f}} & \operatorname{Ker} p & & \end{array} \tag{1}$$

Hence the use of "induced morphism" in *Definition 1.1*.

Example 1.2. A common example of an abelian category is **Ab**, *the category of abelian groups*:
Here, the objects are abelian groups and the morphisms are group homomorphisms. Group homomorphisms naturally satisfy the conditions for Hom **Ab**, and $\circ$ is regular function composition.
The direct sum is the usual direct sum for abelian groups; namely given groups $(A, \bullet)$ and $(B, *)$, the domain of $A \oplus B$ is $A \times B$ and the product is given by

$$(a_1, b_1) \text{❄} (a_2, b_2) = (a_1 \bullet a_2, b_1 * b_2)$$

The zero object in **Ab** is the group $\{0\}$, and the category notions of kernel and cokernel coincide with our usual definitions in the algebraic sense, where for $f : A \to B$

$$\operatorname{Ker} f = \{x \in A : f(x) = 0\} \qquad \text{and} \qquad \operatorname{Coker} f = B/f(A)$$

Finally, as in *Definition 1.1* we wish $\bar{f} : \operatorname{Coker} u \to \operatorname{Ker} p$ to be an isomorphism. Note

$$\operatorname{Coker} u = X/u(\operatorname{Ker} f) = X/\operatorname{Ker} f$$

as u is the canonical embedding. Also since $\operatorname{Ker} p = \{y \in Y : p(y) = 0\}$ and $\operatorname{Coker} f = Y/f(X)$, we have

$$y \in \operatorname{Ker} p \Leftrightarrow p(y) = 0 \text{ (in Coker } f) \ \Leftrightarrow p(y) \in f(X)$$

so $\operatorname{Ker} p = f(X)$. By the first isomorphism theorem for groups, $X/\operatorname{Ker} f \cong f(X)$, thus $\bar{f}$ exists, is unique, and makes (1) commute, as required. ◊

Example 1.3. In a similar way, given a K-algebra A we can show the category of right (or left) A-modules known as Mod$A$ is abelian. ◊

Functor definition. A map to relate two categories in a way that was reminiscent of the properties of homomorphism (preserving identity and operations). In particular, for more structure:

Definition 1.4. Let $T : \mathfrak{C} \to \mathfrak{C}'$ be a functor between categories. T is *additive* if T preserves direct sums and $\forall X, Y \in \operatorname{Ob} \mathfrak{C}$, the map $T_{XY} : \operatorname{Hom}_{\mathfrak{C}}(X, Y) \to \operatorname{Hom}_{\mathfrak{C}'}(T(X), T(Y))$ given by $f \mapsto T(f)$ satisfies $T(f + g) = T(f) + T(g)$ for all $f, g \in \operatorname{Hom}_{\mathfrak{C}}(X, Y)$.

Sequences.

Definition 1.5. Let $\mathfrak{C}$ be an abelian category. A *sequence*

$$\dots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} X_{n+1} \longrightarrow \dots$$

is called *exact* if $\operatorname{Im} f_n = \operatorname{Ker} f_{n+1}$ for all n.
A *short exact sequence* is an exact sequence in $\mathfrak{C}$ of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

Example 1.6. Let N be a normal subgroup of a group G, with identity element 1. Then the following is an exact sequence:

$$1 \xrightarrow{i_1} N \xrightarrow{i_N} G \xrightarrow{\pi} G/N \xrightarrow{0} 1$$

This follows as:

- As i_1, i_N are the standard inclusion maps, $\operatorname{Im} i_1 = 1 = \operatorname{Ker} i_N$.
- As π is the canonical projection mapping, $\operatorname{Im} i_N = N = \operatorname{Ker} \pi$.
- By definition the zero map $0 : G/N \to 1$ sends all elements to 1, so $\operatorname{Im} \pi = \operatorname{Ker} 0$.

◊

Definition 1.7. A short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

splits if there is a map $h : Z \to Y$ such that $g \circ h$ is the identity on Z. This is equivalent to saying $Y \cong X \oplus Z$, by the *Splitting Lemma* ([Hungerford, Theorem 1.18]).

Remark 1.8. Short sequences which split are exact. If

$$0 \longrightarrow X \xrightarrow{f} X \oplus Z \xrightarrow{\pi} Z \longrightarrow 0$$

is a sequence, we can make it exact by setting $\operatorname{Ker} \pi = X$ where π is the projection map. ◊

2. Leading to Derived Functors

Assume all categories are abelian and all functors additive.

Definition 2.1. Let $\mathfrak{C}, \mathfrak{C}'$ be categories and T a functor between them. let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence. We say T is

- *left exact* if
$$0 \to T(X) \xrightarrow{T(f)} T(Y) \xrightarrow{T(g)} T(Z)$$
is exact.
- *right exact* if
$$T(X) \xrightarrow{T(f)} T(Y) \xrightarrow{T(g)} T(Z) \to 0$$
is exact.

Example 2.2. Let $\mathfrak{A}$ be an abelian category, and let $A \in \operatorname{Ob} \mathfrak{A}$. Define

$$F_A(X) = \operatorname{Hom}_{\mathfrak{A}}(A, X) \qquad (\text{also written } \operatorname{Hom}_{\mathfrak{A}}(A, -))$$

This defines a (covariant) left exact functor from $\mathfrak{A}$ to **Ab**[1]. We let F_A act on morphisms in $\operatorname{Hom} \mathfrak{A}$ in the natural way; by composition. Given $f : X \to Y$,

$$F_A(f) = \operatorname{Hom}_{\mathfrak{A}}(A, f) : \operatorname{Hom}_{\mathfrak{A}}(A, X) \to \operatorname{Hom}_{\mathfrak{A}}(A, Y)$$

by

$$F_A(f)(\beta) = f \circ \beta$$

◊

Projective and Injective resolutions.

Definition 2.3. An object I in a category $\mathfrak{A}$ is *injective* if given an injective morphism $f : A \to B \in \operatorname{Hom} \mathfrak{A}$ and a map $\alpha : A \to I$ there exists a map β making the following diagram commute:

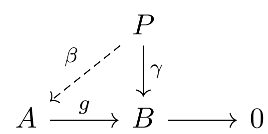
$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \beta} & \\ & & I & & \end{array} \tag{2}$$

One can state the idea of this in simpler terms; "Objects are highly embeddable in I".

[1] N. Jacobson, *Basic Algebra II*, Theorem 3.1.

Definition 2.4. An object P in a category $\mathfrak{A}$ is *projective* if given a surjective morphism $g : A \to B \in \operatorname{Hom} \mathfrak{A}$ and a map $\gamma : P \to B$ there exists a map β making the following diagram commute:

(3)



Again one can state the idea of this in simpler terms; "P is highly projectable".

Remark 2.5. The existence of β in *Definitions 2.3, 2.4* is known as the *universal lifting property.* ♢

Definition 2.6. A category $\mathfrak{A}$ has *enough injectives* if $\forall A \in \operatorname{Ob} \mathfrak{A}$ there is an injective morphism $A \to I$ where I is an injective object.

Similarly

Definition 2.7. A category $\mathfrak{A}$ has *enough projectives* if $\forall A \in \operatorname{Ob} \mathfrak{A}$ there is a surjective morphism $P \to A$ where P is an projective object.

Definition 2.8. Let $M \in \operatorname{Ob} \mathfrak{A}$. An *injective resolution* is an exact sequence of injective modules

$$0 \longrightarrow M \xrightarrow{f} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

Similarly

Definition 2.9. Let $M \in \operatorname{Ob} \mathfrak{A}$. A *projective resolution* is an exact sequence of projective modules

$$\cdots \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

Remark 2.10. If $\mathfrak{A}$ has enough injectives then every object has an injective resolution. Similarly, if $\mathfrak{A}$ has enough projectives then every object has a projective resolution. ♢

A more general idea of *Definitions 2.8, 2.9* is that of a chain complex. We are interested mainly in the abelian category Mod$A$, for A a K-algebra, so we will define complexes (and homologies) in this context:

Definition 2.11. A *chain complex* is a sequence

$$C_\bullet : \ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \ldots \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where the C_i are right A-modules and the d_i are A-homomorphisms such that $d_n \circ d_{n+1} = 0$.

Definition 2.12. A *cochain complex* is a sequence

$$C^\bullet : 0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots$$

where again the C^i are right A-modules and the d^i are A-homomorphisms such that $d^{n+1} \circ d^n = 0$.

As $d_n \circ d_{n+1} = 0$, $\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$. We can then define, for $n \geq 0$, the n^{th} *homology* A-module of $C_\bullet$ (and the n^{th} *cohomology* A-module of $C^\bullet$) to be

$$H_n(C_\bullet) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} \qquad \text{and} \qquad H^n(C^\bullet) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$$

respectively. This provides some sort of measure of how 'nonexact' the complex is.

Definition 2.13. Let $C_\bullet$ and $D_\bullet$ be chain complexes of right A-modules. A *chain morphism* $u : C_\bullet \to D_\bullet$ (commonly referred to as a *morphism*) is a family of A-module homomorphisms $u_n : C_n \to D_n$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle u_{n+1}} & & \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_{n-1}} & & \\ \dots & \longrightarrow & D_{n+1} & \xrightarrow{d'_{n+1}} & D_n & \xrightarrow{d'_n} & D_{n-1} & \longrightarrow & \dots \end{array} \tag{4}$$

Definition 2.14. A chain morphism $u : C_\bullet \to D_\bullet$ is known as a *quasi-isomorphism* (or by Bourbaki as a *homologism*) if the homology maps $h_n : H_n(C_\bullet) \to H_n(D_\bullet)$ are isomorphisms for all n.

Lemma 2.15. *Let $C_\bullet$ be a chain complex of right A-modules. The following are equivalent:*

(1) $C_\bullet$ *is* exact *(that is, exact at every C_n).*
(2) $C_\bullet$ *is* acyclic *(that is, $H_n(C_\bullet) = 0$ for all n).*
(3) *The map $0_\bullet \to C_\bullet$ is a quasi-isomorphism, where $0_\bullet$ is the chain complex of zero modules and zero maps.*

Proof. This follows from the definitions of exact, homology module and quasi-isomorphism:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$$

∎

3. Derived Functors

We now turn to the construction of *right derived functors for left exact sequences* using the tools we have built in the previous sections.
Let $\mathfrak{C}$ be a category with enough injectives, and F a left exact functor. Begin with $M \in \operatorname{Ob} \mathfrak{C}$. As $\mathfrak{C}$ has enough injectives, there is an injective resolution of M

$$0 \longrightarrow M \xrightarrow{f} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

which is a long exact sequence. We rewrite this as two cochain complexes

$$\begin{array}{lccccccccc} M^\bullet : & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \downarrow{\scriptstyle f} & & & & & & \\ I^\bullet : & 0 & \xrightarrow{d^{-1}} & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \longrightarrow & \cdots \end{array}$$

where the two complexes are 'similar' in the sense that their cohomologies at (almost) every point agree. We will work with the $I^\bullet$ cochain complex (in essence suppressing the M object), and we keep in mind it might not be fully exact.
Applying the left exact functor F we obtain

$$F(I^\bullet): 0 \xrightarrow{F(d^{-1})} F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^2) \longrightarrow \cdots$$

We then compute its cohomology at the i^{th} spot and call the resulting object $R^iF(M)$:

$$R^iF(M) := H^i(F(I^\bullet)) = \operatorname{Ker} F(d^i)/\operatorname{Im} F(d^{i-1})$$

and note in particular as F is left exact we have the exact sequence

$$0 \longrightarrow F(M) \xrightarrow{F(f)} F(I^0) \xrightarrow{F(d^0)} F(I^1)$$

and as $f : M \to I^0$ is an injection we conclude

$$R^0F(M) = \operatorname{Ker} F(d^0)/\operatorname{Im} F(d^{-1}) = \operatorname{Im} F(f)/0 \;\cong\; F(M)$$

We say F has been *derived* to form R^iF, namely R^iF is a *derived functor*.

Remark 3.1. In a similar way we can construct the *left derived functors for right exact sequences* by:

(1) Assuming $\mathfrak{C}$ has enough projectives.
(2) Taking a projective resolution of $N \in \operatorname{Ob}\mathfrak{C}$, and forming the chain complex $K_\bullet$.
(3) Computing the i^{th} homology and defining

$$L_iF(N) = H_i(F(K_\bullet)) = \operatorname{Ker} F(d_i)/\operatorname{Im} F(d_{i+1})$$

$\diamondsuit$

Remark 3.2. There is a correspondence between right and left derived functors:

$$R^iF(M) = (L_iF^{op})^{op}(M)$$

which follows from the correspondence between injective resolutions in $\mathfrak{C}$ and projective resolutions in $\mathfrak{C}^{op}$. $\diamondsuit$

We now need to prove that the constructed functor 'fixes' exactness. First, two lemmas which ensure the derived functors are well defined:

Lemma 3.3. *$R^iF(M)$ does not depend on the injective resolution of M. That is, if $J^\bullet$ is a second resolution of M, then*

$$H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$$

Sketch proof. We need the following fact ([Weibel, Theorem 2.3.7]):

If $\epsilon : M \to I^\bullet$ is an injective resolution of M and $f : M \to N$ a map in $\mathfrak{A}$, for every injective resolution $\eta : N \to J^\bullet$ there is a cochain morphism $f' : I^\bullet \to J^\bullet$ 'lifting' f, in the sense $\eta \circ f = f'_0 \circ \epsilon$.

This fact also states f' is unique up to (co)*chain homotopy.* The concept of *homotopy* and its use in this sketch proof is explained in more detail in [Osborne, Chapter 3].
There is thus a cochain morphism $f : I^\bullet \to J^\bullet$ lifting $\mathrm{id}_{\mathfrak{A}}$ yielding the (as it turns out, canonical) map $f^* : H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$. Similarly there is a cochain morphism $g : J^\bullet \to I^\bullet$ lifting $\mathrm{id}_{\mathfrak{A}}$ and the corresponding map g^*. Since $g \circ f$ and $\mathrm{id}_{I^\bullet}$ are cochain morphisms of $I^\bullet \to I^\bullet$ lifting $\mathrm{id}_{\mathfrak{A}}$, we can conclude $g^* \circ f^* = (g \circ f)^* = (\mathrm{id}_{I^\bullet})^* =$ identity on $H^i(F(I^\bullet))$. Similarly $f^* \circ g^* = (\mathrm{id}_{J^\bullet})^*$. Thus f and g are quasi-isomorphisms and

$$H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$$

as required. ∎

Lemma 3.4. *R^iF is a functor. In particular, a morphism $f : M \to N$ yields a morphism $R^iF(f) : R^iF(M) \to R^iF(N)$.*

Sketch proof. If $I^\bullet_M$, $I^\bullet_N$ are injective resolutions of M and N respectively, there is a cochain morphism $f' : I^\bullet_M \to I^\bullet_N$ lifting f unique up to cochain homotopy. Thus by the same ideas as in *Lemma 3.3* the map

$$f'_* : H^i(F(I^\bullet_M)) \to H^i(F(I^\bullet_N))$$

is canonical. This will be the map we want: $R^iF(f) = f'_*$.
The other properties of a covariant functor (preservation of the identity and composition) follow naturally in suit. ∎

We now wish to show right derived functors turn the sequences formed by left exact functors on short exact sequences into long exact sequences.

Theorem 3.5. *Suppose $\mathfrak{C}$ and F are as previously given. Given a short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{5}$$

the following is a long exact sequence:

$$\begin{array}{l} 0 \longrightarrow F(X) \xrightarrow{f^0} F(Y) \xrightarrow{g^0} F(Z) \xrightarrow{h^0} \\ \quad R^1F(X) \xrightarrow{f^1} R^1F(Y) \xrightarrow{g^1} R^1F(Z) \xrightarrow{h^1} \\ \quad R^2F(X) \xrightarrow{f^2} R^2F(Y) \xrightarrow{g^2} \cdots \end{array} \tag{6}$$

Proof. This theorem is proven in the context of left derived functors and projective resolutions, however by *Remark 3.2* we can convert between left and right derived functors and projective and injective resolutions. We need to use two lemmas to prove this statement ([Aluffi, Chapter IX, §7]):

Lemma 3.6. ***Horseshoe Lemma.*** *Given (5) in $\mathfrak{C}$ with $P_X^\bullet, P_Z^\bullet$ the projective resolutions of X, Z respectively, there exists a projective resolution $P_Y^\bullet$ of Y such that*

$$0 \longrightarrow P_X^\bullet \longrightarrow P_Y^\bullet \longrightarrow P_Z^\bullet \longrightarrow 0 \tag{7}$$

is an exact sequence.

Sketch proof. Set $P_Y^i = P_X^i \oplus P_Z^i$. This will form a projective resolution of Y, and all rows of the following diagram will be exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_X^0 & \longrightarrow & P_X^0 \oplus P_Z^0 & \longrightarrow & P_Z^0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_X^1 & \longrightarrow & P_X^1 \oplus P_Z^1 & \longrightarrow & P_Z^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_X^2 & \longrightarrow & P_X^2 \oplus P_Z^2 & \longrightarrow & P_Z^2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{8}$$

■

Lemma 3.7. *Given (7) and F an additive functor of abelian categories, then*

$$0 \longrightarrow F(P_X^\bullet) \longrightarrow F(P_Y^\bullet) \longrightarrow F(P_Z^\bullet) \longrightarrow 0 \tag{9}$$

is an exact sequence.

Proof. Since P_Z^i is projective,

$$\begin{array}{ccccccccc}
 & & & & & \overset{\beta}{\swarrow} & P_Z^i & & \\
 & & & & & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & P_X^i & \xrightarrow{f} & P_Y^i & \xrightarrow{g} & P_Z^i & \longrightarrow & 0
\end{array} \tag{10}$$

there is a map $\beta : P_Z^i \to P_Y^i$ that composes with g to form the identity id - i.e. $\beta = g^{-1}$. Thus P_Y^i can be written as $P_X^i \oplus P_Z^i$, meaning the sequence (10) splits. Then

$$0 \longrightarrow F(P_X^i) \longrightarrow F(P_Y^i) \longrightarrow F(P_Z^i) \longrightarrow 0$$

splits, as F is additive. By *Remark 1.8*, this sequence is exact thus (9) is exact, as required. ■

Returning to the proof of Theorem 3.5*:*
Given (5), we obtain (7) by taking the correct projective resolutions. This is exact by *Lemma 3.6* and by *Lemma 3.7*, (9) is exact. Expanding (9) we obtain:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_X^0) & \longrightarrow & F(P_Y^0) & \longrightarrow & F(P_Z^0) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_X^1) & \longrightarrow & F(P_Y^1) & \longrightarrow & F(P_Z^1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_X^2) & \longrightarrow & F(P_Y^2) & \longrightarrow & F(P_Z^2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{11}
$$

We note that **the homology objects of the column complexes of (11) are exactly the left derived functors of F**, as we defined them.

Finally, by a diagram chase ([Aluffi, Chapter IX, Theorem 3.5]) of these homology objects, we obtain the long exact sequence:

$$
\cdots \longrightarrow L_2F(X) \xrightarrow{f^2} L_2F(Y) \xrightarrow{g^2} L_2F(Z) \xrightarrow{h^2} L_1F(X) \xrightarrow{f^1} L_1F(Y) \xrightarrow{g^1} L_1F(Z) \xrightarrow{h^1} F(X) \xrightarrow{f^0} F(Y) \xrightarrow{g^0} F(Z) \longrightarrow 0
$$

using the *Snake Lemma*. Thus, based on these results, we can conclude (6) is a long exact exact sequence too, as required. ∎

Lemma 3.8. *If A is injective then $R^iF(A) = 0$ for $i \geq 1$.*

Proof. If A is injective, it has an injective resolution

$$X^\bullet : 0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

so

$$R^iF(A) = H^i(F(X^\bullet)) = 0$$

for $i \geq 1$, by induction on the length of $X^\bullet$. ∎

Corollary 3.9. *If A is projective then $L_iF(A) = 0$ for $i \geq 1$.* ∎

Remark 3.10. The famous *Snake Lemma* can be viewed as a 'special case' of *Theorem 3.5* - it should be noted this is *not* a proof of the Snake Lemma, rather a way to rewrite it to demonstrate the previous theorem. Recall the Snake Lemma says for a commutative diagram of exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L' & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & \longrightarrow & 0 \end{array} \tag{12}$$

there is a connecting homomorphism δ

$$\begin{array}{ccccccccc} & & \mathrm{Ker} f & \xrightarrow{u} & \mathrm{Ker} g & \xrightarrow{v} & \mathrm{Ker} h & \overset{\delta}{\dashrightarrow} & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L' & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & \dashrightarrow & \mathrm{Coker} f & \xrightarrow{u'} & \mathrm{Coker} g & \xrightarrow{v'} & \mathrm{Coker} h & & \end{array} \tag{13}$$

making the sequence

$$0 \longrightarrow \mathrm{Ker} f \xrightarrow{u} \mathrm{Ker} g \xrightarrow{v} \mathrm{Ker} h \xrightarrow{\delta} \mathrm{Coker} f \xrightarrow{u'} \mathrm{Coker} g \xrightarrow{v'} \mathrm{Coker} h \longrightarrow 0$$

exact. If we view the columns of (12) as complexes

$$L^\bullet :\ 0 \longrightarrow L \xrightarrow{f} L' \longrightarrow 0 \longrightarrow \cdots$$

$$M^\bullet :0 \longrightarrow M \xrightarrow{g} M' \longrightarrow 0 \longrightarrow \cdots$$

$$N^\bullet :\ 0 \longrightarrow N \xrightarrow{h} N' \longrightarrow 0 \longrightarrow \cdots$$

then (12) is the expansion of the short exact sequence

$$0 \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet \longrightarrow 0$$

of these complexes. The Snake Lemma then tells us there is an exact sequence

$$0 \longrightarrow H^0(L^\bullet) \longrightarrow H^0(M^\bullet) \longrightarrow H^0(N^\bullet) \xrightarrow{\delta} H^1(L^\bullet) \longrightarrow H^1(M^\bullet) \longrightarrow H^1(N^\bullet) \longrightarrow 0$$

which we can see is a special case of (6). ♢

4. Applications: the Ext Functor

A popular example of derived functors are the Hom functors obtained by deriving the Hom functors. We have already come across Hom functors before; in *Example 2.2* we defined the Hom functor on abelian categories:

$$\mathrm{Hom}_{\mathfrak{A}}(A,-):\mathfrak{A}\to\mathbf{Ab}\qquad X\mapsto\mathrm{Hom}_{\mathfrak{A}}(A,X)$$

and found it was a left exact functor. Using the previous results we obtain the i^{th} right derived functor of $\mathrm{Hom}_{\mathfrak{A}}(A,-)$ as

$$\mathrm{Ext}^i_{\mathfrak{A}}(A,-):=R^i\,\mathrm{Hom}_{\mathfrak{A}}(A,-)$$

Similarly
If A is a K-algebra we can similarly define the functor $\mathrm{Hom}_K(M,-)$ over $\mathrm{Mod}A$ and obtain the n^{th} derived functor

$$\mathrm{Ext}^n_K(M,-):\mathrm{Mod}A\to\mathrm{Mod}K$$

of $\mathrm{Hom}_K(M,-)$ (for a fixed A-module M).

Example 4.1. We will compute $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})$.
First, take a projective resolution of $\mathbb{Z}_p$:

$$0\longrightarrow\mathbb{Z}\xrightarrow{\times p}\mathbb{Z}\xrightarrow{\equiv_p}\mathbb{Z}_p\longrightarrow 0$$

Applying the right exact contravariant functor[2] $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$, we get the exact sequence

$$0\longrightarrow\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})\xrightarrow{(\times p)^*}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})\xrightarrow{(\equiv_p)^*}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})$$

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})\cong\mathbb{Z}$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})=0$ we get the following:

$$0\longrightarrow\mathbb{Z}\xrightarrow{(\times p)^*}\mathbb{Z}\xrightarrow{(\equiv_p)^*}0$$

Finally

- $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})=\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})=0$
- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})=\mathrm{Ker}(\times p)^*/\,\mathrm{Im}(\equiv_p)^*=\mathbb{Z}_p/0=\mathbb{Z}_p$
- $\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})=0$ for $i\geq 2$.

Note that by *Corollary 3.9* this means $\mathbb{Z}_p$ isn't projective as a $\mathbb{Z}$-module. ♢

Remark 4.2. If G is a finitely generated abelian group, then $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z})\cong G$. This follows from the fundamental theorem of finitely generated abelian groups, which says G can be written as a direct sum involving $\mathbb{Z}^n$ and $\mathbb{Z}_p$. ♢

[2] Applying a contravariant functor to a projective resolution also leads to a right derived functor.

Remark 4.3. By *Lemma 3.8* if N is an injective module we can immediately conclude

$$\forall M,\ \mathrm{Ext}^n_K(M,-)(N) = \mathrm{Ext}^n_K(M,N) = 0$$

for all $n \geq 1$. $\diamondsuit$

5. Closing remarks

Definition 5.1. A (covariant) *cohomological δ-functor* between abelian categories $\mathfrak{A}$ and $\mathfrak{B}$ is a collection of additive functors $T^n : \mathfrak{A} \to \mathfrak{B}$ (indexed by nonnegative integers) together with a family of morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ for each short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{14}$$

in $\mathfrak{A}$, such that the following two properties hold:

(1) For each short exact sequence (14) there is a long exact sequence

$$0 \longrightarrow T^0(A) \longrightarrow T^0(B) \longrightarrow T^0(C) \xrightarrow{\delta^0}$$
$$\longrightarrow T^1(A) \longrightarrow \cdots \longrightarrow T^n(C) \xrightarrow{\delta^n}$$
$$\longrightarrow T^{n+1}(A) \longrightarrow T^{n+1}(B) \longrightarrow \cdots$$

(2) Each morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \downarrow & & \downarrow \\ T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A') \end{array}$$

(This property is known as *naturality*.)

This definition generalises the notion of right derived functors; by *Theorem 3.5* they satisfy (1) and their naturality is proven in [Weibel, Theorem 2.4.6][3].
Similarly the concept of a *homological* δ-functor can be defined to generalise left derived functors.

[3] For left derived functors.

Introduced by Grothendieck in his famous *Tôhoku paper*, this context is intended to be the appropriate setting in which to treat and further the development of derived functors.

We have thus plotted the course of derived functors from their roots in categories and exact sequences to the point where we see they are part of a much larger, more general framework, and here we end.

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