An Analysis of Tame Topology using O-Minimality

Brian Tyrrell, Trinity College Dublin. Supervised by Professor Andreea Nicoara.

1. Introduction

Grothendieck first presented the idea of *tame topology* in 1984 in *Equisse d'un Programme*, where he called for the recasting of topology to reflect the 'moderate' or 'tame' topological properties of the semialgebraic sets (§3) and avoid unpleasant results, like the Banach-Tarski Paradox, or unpleasant objects, like the topologist's sine curve or space-filling curves. The first collection to generalise the properties of the semialgebraic sets was the subanalytic sets, and it was discovered the study of *o-minimal structures* is a further generalisation.

O-minimality (short for *order-minimality*) originally arose in the 1980's through the work of van den Dries [2], and Knight, Pillay and Steinhorn [4, 5] in *model theory*. Model theory is a branch of mathematical logic concerned with studying mathematical structures by examining what is true (from a logical perspective) in these structures, and what subsets of these structures can be defined by first order logical formulae.

The principal objects of study are *models*, also known as *structures*:

Definition 1.1. Let L be a collection of *constants*, *relations*, and *functions*, known as a *language*. An *L-structure* $\mathfrak{A}$ is the data of:

- An underlying set, denoted $|\mathfrak{A}|$.
- *Interpretations* for all symbols in L, meaning:
  - Each constant symbol $c \in L$ is assigned to an element $c^{\mathfrak{A}}$ of $|\mathfrak{A}|$.
  - Each relation symbol $R \in L$ of arity $k < \omega$ is interpreted to hold on some subset $R \subseteq |\mathfrak{A}|^k$, meaning
$$R^{\mathfrak{A}}(x_1, \dots, x_k) \text{ is true } \Leftrightarrow (x_1, \dots, x_k) \in R.$$
  - Each function symbol $f \in L$ of arity $k < \omega$ is interpreted to take every element of $|\mathfrak{A}|^k$ to an element of $|\mathfrak{A}|$. That is, f is a function
$$f^{\mathfrak{A}} : |\mathfrak{A}|^k \to |\mathfrak{A}|.$$

Example 1.2. Let $\mathfrak{N} = (\mathbb{N}, 0, <, +, \times, S)$. This structure is the *standard model of arithmetic*: we assign the usual interpretation to each element of the language. In terms of the previous definition, for example:

$$<^{\mathfrak{N}} \subset \mathbb{N}^2 \qquad <^{\mathfrak{N}} = \{(1,2), (1,3), (2,3), \dots\}.$$

Using the definition of a *structure* we can make advances towards answering Grothendieck's challenge to describe a general class of sets that shares the tame topological properties of the semialgebraic sets.

2. O-Minimality

In first order logic we can express ourselves through *formulae* and *sentences* using the relations, functions, and constants of our language L, connectives such as $\wedge$ (and), $\vee$ (or), $\to$ (implies), quantifiers $\exists$ (there exists), $\forall$ (for all), and parentheses '(', ')'. If a sentence φ is true in a structure $\mathfrak{A}$ this is written $\mathfrak{A} \models \varphi$.

Example 2.1. Consider $\mathfrak{N}$, the *standard model of arithmetic* (*Example 1.2*). Then

$$\mathfrak{N} \models \forall x \; ((0 < x) \vee (x = 0)).$$

Formulae (such as $\varphi(x) = 0 < x$) can define a subset A of a structure $\mathfrak{A}$ by

$$a \in A \qquad \Leftrightarrow \qquad \mathfrak{A} \models \varphi(a).$$

Sets defined by formulae like this are known as *definable sets*. **NB:** A function is called *definable* if its graph is a definable set. The definable sets of o-minimal structures are particularly simple:

Definition 2.2. An L-structure $\mathfrak{A}$ is said to be *o-minimal* if every definable subset of $\mathfrak{A}$ is a finite union of singletons and open intervals.

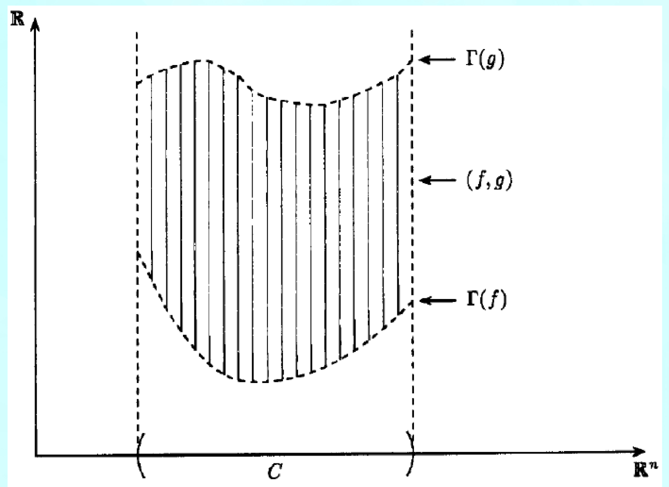
This condition has profound repercussions for definable subsets in *any* dimension, and for the (interval) topology on the space: given a definable set $A \subseteq |\mathfrak{A}|^m$, some $m \geq 1$, we can partition A into a disjoint union of *cells*:

Definition 2.3. An *cell* is a definable subset of $|\mathfrak{A}|^m$ obtained by induction as follows:

(1) A point $\{a\} \subseteq |\mathfrak{A}|$ or an interval $(a, b) \subseteq |\mathfrak{A}|$ is a cell.

(2) If X is a cell, the graph of a definable, continuous function $f : X \to |\mathfrak{A}|$ is a cell.

(3) If X is a cell, for $f < g$ definable and continuous functions, $\{(x, a) \in X \times |\mathfrak{A}| : f(x) < a < g(x)\}$ is a cell.



Examples of cells in $\mathbb{R}^{n+1}$ [3].

Theorem 2.4. Cell Decomposition theorem.

$(\text{I})_{\text{m}}$ *Given definable sets $A_1, \dots, A_k \subseteq |\mathfrak{A}|^m$, there is a decomposition of $|\mathfrak{A}|^m$ into cells, partitioning each of the $A_1, \dots, A_k$.*

$(\text{II})_{\text{m}}$ *Suppose $A \subseteq |\mathfrak{A}|^m$ is definable. Given a definable function $f : A \to |\mathfrak{A}|$, there is a decomposition of $|\mathfrak{A}|^m$ partitioning A such that $f|_B$ for each cell B in the decomposition with $B \subseteq A$ is continuous.*

3. Tame Topology

Of tame topology, A'Campo, Ji and Papadopoulos [1] write:

> Grothendieck recalls that the field of topology at the time he wrote his Esquisse was still dominated by the development, done during the 1930s and 1940s, by analysts, in a way that fits their needs, rather than by geometers. He writes that the problem with such a development is that one has to deal with several pathological situations that have nothing to do with geometry. He declares that the fact that "the foundations of topology are inadequate is manifest from the very beginning, in the form of 'false problems' (at least from the point of view of the topological intuition of shape)."

There is no strict definition to what constitutes calling a property *tame*; rather as this idea arose from observing the nice topological and geometric properties of the semialgebraic sets, we observe what properties are common to these sets which constitute preferable and advantageous behaviour. Properties such as:

(1) **Stratification.** If X is semialgebraic, then X can be written as a disjoint union $X = X_1 \cup \cdots \cup X_n$ of semialgebraic sets, where each X_i is a connected real analytic manifold.

(2) **Smooth maps.** If $f : X \to \mathbb{R}$ is semialgebraic, then X can be partitioned into finitely many disjoint semialgebraic sets X_i such that $f|_{X_i}$ is analytic.

We can see properties (1) & (2) correspond to $(\text{I})_{\text{m}}$ & $(\text{II})_{\text{m}}$ in *Theorem 2.4*. As it turns out, in specific structures $(\text{I})_{\text{m}}$ & $(\text{II})_{\text{m}}$ can be extended to include analytic functions. Specific o-minimal structures also have a host of other tame properties, such as definable Skolem functions, definable partitions of unity, a uniform bound on the size of fibers of definable sets, and a set being definably connected if and only if it is definably path connected. In general the projection of a definable set to a lower dimension is also definable.

4. References

[1] A'Campo, N., Ji L. and Papadopoulos A. *On Grothendieck's Tame Toplogy*, `arXiv:1603.03016 [math.GT]`, submitted 9/03/2016

[2] van den Dries, L. *Remarks on Tarski's Problem Concerning* $(\mathbb{R}, +, \cdot, \exp)$, Logic Colloquium '82, G. Lolli, G. Longo, A. Marja, eds., North Holland, pp. 97-121, 1984

[3] van den Dries, L. *Tame topology and o-minimal structures*, Cambridge University Press, 1998

[4] Knight, J.F., Pillay, A. and Steinhorn, C. *Definable sets in ordered structures. II*, Trans. Amer. Math. Soc. **295**(2), pp.593-605, 1986

[5] Pillay, A. and Steinhorn, C. *Definable sets in ordered structures. I*, Trans. Amer. Math. Soc. **295**(2), pp. 565-592, 1986

[Poster] Amberg, B. `http://www.brian-amberg.de/uni/poster/`