Conditions a countable model must satisfy for its Scott sentence to have an uncountable model of a specified cardinality

"Models Wanted" MathSoc talk

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- You'll see me using ω and ω_1 - think of this as saying "I'm counting to $\aleph_0$ or $\aleph_1$" respectively.
- My project was focused on sets of cardinality $\aleph_1$ and $\aleph_2$ – the next cardinals after $\aleph_0$.

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This object is also referred to as a *model*.

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- This model has rules in place so it behaves like the Natural numbers we're used to. Rules like:
- "0 is the smallest element"
- "1 comes before 2"
- "I can always find a bigger element"
- How do we write these rules and how do we ensure the model obeys them?

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Sentences

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- For example, in the model $\mathfrak{A}$ I can turn the sentence "0 is the smallest element" into

$$\neg\exists x(x < 0) \qquad \text{or} \qquad \forall x(x = 0 \vee 0 < x)$$

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- (For example, our model of the Natural numbers satisfies the sentence "$0 = 0$" but not "$0 = 1$")
- If the sentence is false in the model, then it is not satisfied, written $\mathfrak{A} \not\models \varphi$.
- However from the definition of truth,

$$\mathfrak{A} \not\models \varphi \Leftrightarrow \mathfrak{A} \models \neg\varphi$$

Scott sentences

- In 1965 Scott proved by construction for any countable model there existed a sentence that could describe the model completely; he constructed, for a given countable structure $\mathfrak{A}$, a sentence φ such that if $\mathfrak{B} \models \varphi$ (where $\mathfrak{B}$ is another countable structure) then $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$ (written $\mathfrak{B} \cong \mathfrak{A}$).

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- In other words, Scott sentences are sentences of the logic $L_{\omega_1,\omega}$.

Example

A vector space of dimension n
Let $\mathfrak{A} = (V, 0, +, -, (*)_{q\in\mathbb{Q}})$ be a model.
To say the dimension is *at least n*;

$$\psi(x_1, \ldots, x_n) = \bigwedge_{q_1,\ldots,q_n\in\mathbb{Q}} q_1 * x_1 + \cdots + q_n * x_n = 0 \leftrightarrow (q_1 = 0 \wedge \cdots \wedge q_n = 0)$$

Which is to say "$x_1, \ldots, x_n$ are linearly independent". To say the dimension is *at most n*;

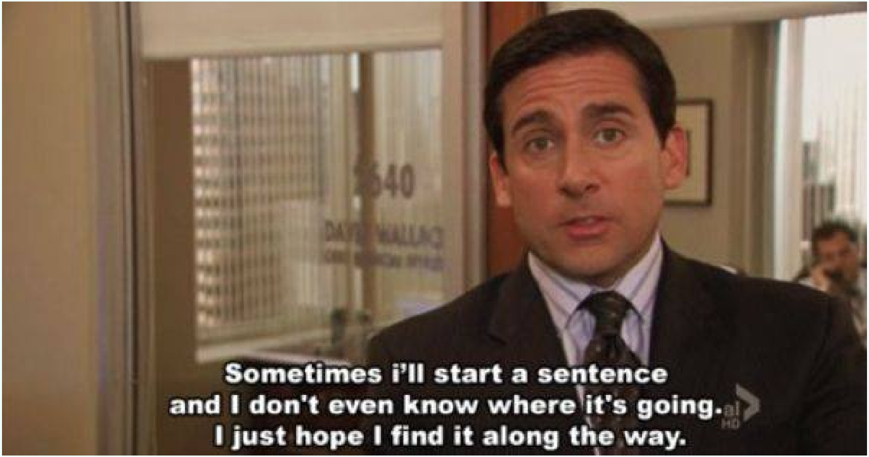
$$\xi(x_1, \ldots, x_n) = \quad \forall y \left(\bigvee_{q_1,\ldots,q_n\in\mathbb{Q}} y = q_1 * x_1 + \cdots + q_n * x_n \right)$$

Which is to say "$x_1, \ldots, x_n$ span the space". Suppose ϕ captures the axioms of a vector space; all together,

$$\phi \wedge \exists x_1, \ldots, x_n(\psi(x_1, \ldots, x_n) \wedge \xi(x_1, \ldots, x_n))$$

forms a Scott sentence.

A *Michael Scott* sentence:



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DEFINITION

A partial isomorphism between (countable) structures $\mathfrak{A},\mathfrak{B}$ is a bijection $f : U \to V$ on subsets U, V of $\mathfrak{A}$, $\mathfrak{B}$ which itself is an isomorphism.

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A back-and-forth system P is a nonempty set of partial isomorphisms $f : U \to V$ with the properties that

1. For each $f \in P$ and $x \in \mathfrak{A}$ there is a $y \in \mathfrak{B}$ and $f^+ \in P$ such that $f^+ : U \cup \{x\} \to V \cup \{y\}$ and $f^+(x) = y$.
2. For each $f \in P$ and $y \in \mathfrak{B}$ there is an $x \in \mathfrak{A}$ and $f^+ \in P$ such that $f^+ : U \cup \{x\} \to V \cup \{y\}$ and $f^+(x) = y$.

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Theorem

If there exists a a back-and-forth system P on two countable structures $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$. ■

Types

Definition

Let $\mathfrak{A}$ be a structure with language L. For $\vec{a} = (a_1, \ldots, a_n) \in \mathfrak{A}$, the type of $\vec{a}$ (denoted $\text{tp}(\vec{a})$) is the set of all formulas $\varphi(\vec{x})$ with $\mathfrak{A} \models \varphi(\vec{a})$. Furthermore,

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Definition

A complete type $P(\vec{x})$ in variables $\vec{x} = (x_1, \ldots, x_n)$ contains $\varphi(\vec{x})$ or $\neg\varphi(\vec{x})$ for every elementary first order formula $\varphi(\vec{x})$ in the variables $x_1, \ldots, x_n$.

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A countable structure $\mathfrak{A}$ is (ω-)*homogeneous* if for any $\vec{a}$, $\vec{b} \in \mathfrak{A}$ s.t. $\vec{a}, \vec{b}$ satisfy the same formulas there is an automorphism of $\mathfrak{A}$ taking $\vec{a}$ to $\vec{b}$.

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Definition

Let $(\mathfrak{A}, U)$ be a pair where $\mathfrak{A}$ is a countable structure and U is a predicate. $(\mathfrak{A}, U)$ is *pair*-homogeneous if, given $\vec{a}, \vec{b}, c$ such that $\vec{a}$ and $\vec{b}$ realise the same type in $(\mathfrak{A}, U)$, there exists $d \in \mathfrak{A}$ such that $(\vec{a}, c)$ and $(\vec{b}, d)$ realise the same type in $(\mathfrak{A}, U)$.

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Remark

Note: pair-homogeneity + back-and-forth system = homogeneous.

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DEFINITION

Given an L-structure $\mathfrak{A}$,

$$\mathrm{Th}(\mathfrak{A}) = \{\varphi : \varphi \text{ is a sentence of } L \text{ and } \mathfrak{A} \models \varphi\}.$$

This is known as the *theory* of $\mathfrak{A}$.

Atomic models

Remark

Note that if $\mathfrak{A}$ is a countable atomic model, then it has a Scott sentence φ that is the conjunction of $\mathrm{Th}(\mathfrak{A})$ and a sentence saying

$$\forall \vec{x} \left(\bigvee_i \gamma_i(\vec{x}) \right)$$

where the γ_i are the generators for the complete atomic types consistent with $\mathrm{Th}(\mathfrak{A})$.

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Theorem

Suppose $\mathfrak{A}$, $\mathfrak{B}$ are atomic models for the same theory, where $\mathfrak{A}$ is countable. Then $\mathfrak{B}$ satisfies the Scott sentence of $\mathfrak{A}$.

Proof

Show the two structures satisfy the *same* types. ■

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What to take away from this: if $\mathfrak{B} \models \varphi$ then there is a countable substructure $\mathfrak{B}_0$ of $\mathfrak{B}$ satisfying φ.

Countable fragments

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We proceed as follows: let F be a countable fragment of $L_{\omega_1,\omega}$ including φ and be closed under subformulas of φ, and include all finitary formulas of L and be closed under $\wedge, \vee, \neg$ (note the language of $\mathfrak{A}$ is *countable*).
Using the Infinitary Downward Löwenheim Skolem Tarski Theorem we can obtain a countable $\mathfrak{B}_0$ which satisfies φ. Thus $\mathfrak{B}_0 \cong \mathfrak{A}$.

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Theorem

Suppose $\mathfrak{A}$ is a countable model and has an expansion $\mathfrak{A}^* = (\mathfrak{A}, U^{\mathfrak{A}^*})$ satisfying T^*. The substructure formed by restricting $\mathfrak{A}^*$ to $U^{\mathfrak{A}^*}$ is isomorphic to $\mathfrak{A}$.

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Theorem

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Proof

First, D_ω is homogeneous.
D_ω realises the same types as D_0.
D_ω is still countable.
Thus we have two countable, homogeneous structures that realise the same types, so they are isomorphic, as required. ■

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Main Theorem

Now we can begin one of the main results.
Recall T^* is a set of (finite) sentences in the language of $\mathfrak{A}$ with an added symbol U, saying the following:

(A) $(\exists x)\neg Ux$

(B) For each formula $\phi(\vec{u}, x)$, " $\forall \vec{u} \in U$ "

$$\exists x(\phi(\vec{u}, x)) \rightarrow \exists x(Ux \wedge \phi(\vec{u}, x))$$

As it turns out, these are the conditions we need for a Scott sentence of a countable structure to be satisfied in an $\aleph_1$-sized structure.

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Set $D_0 = \mathfrak{B}$ and $D_1 = \mathfrak{A}$ and construct the chain $(D_\alpha)_{\alpha<\omega_1}$; where at the right places, $D_\gamma = \bigcup_{\beta<\gamma} D_\beta$ and $(D_{\alpha+1}, D_\alpha) \cong (D_1, D_0)$. Note for all $\alpha < \omega_1$, $D_\alpha \cong D_0$.

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Set $\mathfrak{M} = \bigcup_{\alpha<\omega_1} D_\alpha$ which has cardinality $\aleph_1$. $\mathfrak{M}$ has 'nice properties': it is a model of $\mathrm{Th}(\mathfrak{A})$ and is atomic (exercise left to the reader).

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Therefore since we know what these Scott sentences look like, $\mathfrak{M} \models \varphi$, as required. ■

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3. The $\aleph_2$-proof is presented in a 'step-by-step' way so there's an indication on how to generalise to $\aleph_\alpha$, but more work is needed!
4. Scattered about the place are other small theorems with stricter conditions that guarantee models of any size (in essence, solve the problem in general).

Conclusion

The paper in full can be found at www.maths.tcd.ie/∼btyrrel/REUs.html .

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Thank you for listening!
Questions?