HOMOLOGICAL ALGEBRA

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Abstract. In this report we will assemble the pieces of homological algebra needed to explore derived functors from their base in exact sequences of abelian categories to their realisation as a type of δ-functor, first introduced in 1957 by Grothendieck. We also speak briefly on the typical example of a derived functor, the Ext functor, and note some of its properties.

Contents

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1. Introduction

We will begin by defining the notion of a *category*;

Definition 1.1. A *category* is a triple $\mathfrak{C} = (\operatorname{Ob}\mathfrak{C}, \operatorname{Hom}\mathfrak{C}, \circ)$ where

- $\operatorname{Ob}\mathfrak{C}$ is the *class of objects* of $\mathfrak{C}$.
- $\operatorname{Hom}\mathfrak{C}$ is the *class of morphisms* of $\mathfrak{C}$.

Furthermore, $\forall X, Y \in \operatorname{Ob}\mathfrak{C}$ we associate a set $\operatorname{Hom}_{\mathfrak{C}}(X, Y)$ - the *set of morphisms from X to Y* - such that $(X, Y) \neq (Z, U) \Rightarrow \operatorname{Hom}_{\mathfrak{C}}(X, Y) \cap \operatorname{Hom}_{\mathfrak{C}}(Z, U) = \emptyset$.
Finally, we require $\forall X, Y, Z \in \operatorname{Ob}\mathfrak{C}$ the operation

$$\circ : \operatorname{Hom}_{\mathfrak{C}}(Y, Z) \times \operatorname{Hom}_{\mathfrak{C}}(X, Y) \to \operatorname{Hom}_{\mathfrak{C}}(X, Z) \qquad (g, f) \mapsto g \circ f$$

to be defined, associative and for all objects the identity morphism must exist, that is, $\forall X \in \operatorname{Ob}\mathfrak{C}\ \exists 1_X \in \operatorname{Hom}_{\mathfrak{C}}(X, X)$ such that $\forall f \in \operatorname{Hom}_{\mathfrak{C}}(X, Y),\ g \in \operatorname{Hom}_{\mathfrak{C}}(Z, X),\ f \circ 1_X = f$ and $1_X \circ g = g$.

This is a very general definition for a structure; so much so that they were called "abstract nonsense" by Norman Steenrod. Aluffi in [1] retorts "[abstract nonsense] is essentially accurate and not necessarily derogatory: categories refer to nonsense in the sense that they are all about the 'structure', and not about the 'meaning', of what they represent." The author is forced to agree with both viewpoints: by defining a very broad structure, we can encompass many algebraic objects and instead focus on how they relate to other algebraic objects, however for the notion of category to be useful additional properties need to be introduced. *Abelian categories* (§2) have these additional properties and are commonly used as a basic object in this branch of mathematics.

If we wish to relate two categories, we can do so via a *functor*;

Definition 1.2. Let $\mathfrak{C}, \mathfrak{C}'$ be categories. A *covariant functor* $T : \mathfrak{C} \to \mathfrak{C}'$ is a map allocating every object X of $\mathfrak{C}$ to an object $T(X)$ of $\mathfrak{C}'$, every morphism $f : X \to Y$ in $\mathfrak{C}$ to a morphism $T(h) : T(X) \to T(Y)$ in $\mathfrak{C}'$ such that;

1. T preserves identity, that is, $\forall X \in \operatorname{Ob}\mathfrak{C}$, $T(1_X) = 1_{T(X)}$ [1].
2. T preserves $\circ$, namely for all pairs of morphisms $f : X \to Y$, $g : Y \to Z$ of $\mathfrak{C}$, $T(g \circ f) = T(g) \circ T(f)$.

Note these properties (preserving identities and operations) are reminiscent of a homomorphism (between groups, modules, etc) of algebraic structures.

We can then naturally wonder about sequences of abelian categories - in particular we discover in §2.3 sequences that are particularly 'nice'; exact sequences. When we apply functors to these exact sequences, however, exactness can fail, which leads us to *derived* functors via §3:

If a functor fails to preserve the exactness of a sequence, it can be derived in an attempt to repair the failure of exactness. In some sense these derived

[1] 1_X is the identity function on X.

functors measure to what extent the original functor fails to be exact. The construction of derived functors is covered in §5 and the Ext functor, (realised as the derivation of a Hom functor) is outlined in §5.1. Homological algebra arose in part from the study of Ext on abelian groups, thus derived functors are a cornerstone of the subject.

Chain homotopies are defined in §4 and two lemmas important to the construction of derived functors are also proven in this section. Finally in §6 we leave the reader with a glimpse to the abstraction of a derived functor by defining a *δ-functor*.

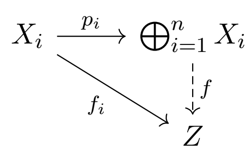
2. Background & Opening Definitions

To begin, we recall some basic definitions regarding category theory. The appendix of [2] holds many key ideas presented here, and the author also recommends [8] for more detailed accounts.

2.1. **Categories.**

We wish to impose more order on the very general structure that is a category. We first define:

Definition 2.1. A *direct sum* of objects $X_1, \dots, X_n$ in $\mathfrak{C}$ is an object $\bigoplus_{i=1}^n X_i := X_1 \bigoplus \cdots \bigoplus X_n$ of $\mathfrak{C}$ with morphisms $p_i : X_i \to \bigoplus_{i=1}^n X_i$ such that for all objects $Z \in \mathfrak{C}$ and morphisms $f_i : X_i \to Z$ there exists a unique morphism $f : \bigoplus_{i=1}^n X_i \to Z$ making the following diagram commute:

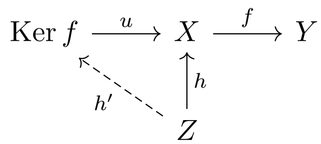


Definition 2.2. Let $\mathfrak{C}$ be a category. $\mathfrak{C}$ is *additive* if:

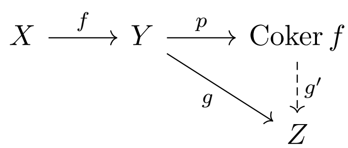
- For any finite set of objects $X_1, \dots, X_n$ there exists a direct sum $\bigoplus_{i=1}^n X_i$ in $\mathfrak{C}$.
- $\forall X, Y \in \operatorname{Ob}\mathfrak{C}$, $\operatorname{Hom}_{\mathfrak{C}}(X,Y)$ is equipped with an abelian group structure.
- $\circ$ is bilinear.
- The *zero object* exists, that is, an object $0 \in \operatorname{Ob}\mathfrak{C}$ such that 1_0 is the zero element in $\operatorname{Hom}_{\mathfrak{C}}(0,0)$.

We also wish to introduce aspects of algebra such as the Kernel or Cokernel of a map, however we don't have access to elements of objects, so we proceed a little more cautiously:

Definition 2.3. Let $\mathfrak{C}$ be an additive category and $f : X \to Y$ a morphism of $\mathfrak{C}$. A *kernel* of f is an object $\operatorname{Ker} f$ and a morphism $u : \operatorname{Ker} f \to X$ such that $f \circ u = 0$ and $\forall Z \in \operatorname{Ob}\mathfrak{C}$, for all morphisms $h : Z \to X \in \operatorname{Hom}\mathfrak{C}$ such that $f \circ h = 0$ there exists a unique morphism $h' : Z \to \operatorname{Ker} f$ such that the following diagram commutes:



The *cokernel* of f, $\operatorname{Coker} f$, is defined analogously resulting in the following commutative diagram:



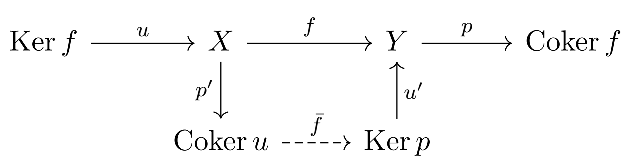
where $p \circ f = 0$.

Definition 2.4. A morphism $u : X \to Y \in \operatorname{Hom} \mathfrak{C}$ is an *isomorphism* if there exists a morphism $v : Y \to X \in \operatorname{Hom} \mathfrak{C}$ such that $uv = 1_Y$ and $vu = 1_X$.

Definition 2.5. A category $\mathfrak{C}$ is *abelian* if $\mathfrak{C}$ is additive and if each morphism $f \in \operatorname{Hom} \mathfrak{C}$ admits a kernel $u : \operatorname{Ker} f \to X$ and cokernel $p : Y \to \operatorname{Coker} f$ and the induced morphism, $\bar{f} : \operatorname{Coker} u \to \operatorname{Ker} p$, is an isomorphism.

Note that if every morphism in a category $\mathfrak{C}$ admits a kernel and cokernel, $\bar{f}$ naturally exists, is unique, and makes the following diagram commute:

(1)



(See [2, Appendix A.1].) Hence the use of "induced morphism" in *Definition 2.5*.

Example 2.6. A common example of an abelian category is **Ab**, *the category of abelian groups*:
Here, the objects are abelian groups and the morphisms are group homomorphisms. Group homomorphisms naturally satisfy the conditions for Hom **Ab**, and $\circ$ is regular function composition.
The direct sum is the usual direct sum for abelian groups; namely given groups $(A, \bullet)$ and $(B, *)$, the domain of $A \oplus B$ is $A \times B$ and the product is given by

$$(a_1, b_1) \circledast (a_2, b_2) = (a_1 \bullet a_2, b_1 * b_2)$$

The zero object in **Ab** is the group $\{0\}$, and the category notions of kernel and cokernel coincide with our usual definitions in the algebraic sense, where for $f : A \to B$

$$\operatorname{Ker} f = \{x \in A : f(x) = 0\} \qquad \text{and} \qquad \operatorname{Coker} f = B/f(A)$$

Finally, as in *Definition 2.5* we wish $\bar{f} : \operatorname{Coker} u \to \operatorname{Ker} p$ to be an isomorphism. Note

$$\operatorname{Coker} u = X/u(\operatorname{Ker} f) = X/\operatorname{Ker} f$$

as u is the canonical embedding. Also since $\operatorname{Ker} p = \{y \in Y : p(y) = 0\}$ and $\operatorname{Coker} f = Y/f(X)$, we have

$$y \in \operatorname{Ker} p \Leftrightarrow p(y) = 0 \text{ (in } \operatorname{Coker} f) \Leftrightarrow p(y) \in f(X)$$

so $\operatorname{Ker} p = f(X)$. By the first isomorphism theorem for groups, $X/\operatorname{Ker} f \cong f(X)$, thus $\bar{f}$ exists, is unique, and makes (1) commute, as required. $\diamondsuit$

Example 2.7. In a similar way, given a K-algebra A we can show the category of right (or left) A-modules, known as Mod$A$, is abelian.
The objects are right A-modules and the morphisms are A-module homomorphisms. The direct sum is the standard K-vector space direct sum with the additional structure of a right A-module induced by the natural scalar multiplication. The zero object is the trivial A-module $\{0\}$ and the kernel and cokernel coincide with our standard conception; for an A-module homomorphism $f : M \to N$

$$\operatorname{Ker} f = \{x \in M : f(x) = 0\} \qquad \text{and} \qquad \operatorname{Coker} f = N/f(M)$$

Finally by the first isomorphism theorem for modules (as in *Example 2.6*) we obtain the required isomorphism $\bar{f} : \operatorname{Coker} u \to \operatorname{Ker} p$ for (1), making Mod$A$ an abelian category. $\diamondsuit$

Finally if we want to introduce the possibility of scalars into our category structure we define what might be called a *category over a field*:

Definition 2.8. Let K be a field. A K-category is a category $\mathfrak{C}$ such that for all pairs $X, Y \in \operatorname{Ob}\mathfrak{C}$, $\operatorname{Hom}_{\mathfrak{C}}(X, Y)$ is equipped with a K-vector space structure such that $\circ$ is a K-bilinear map.

2.2. **Functors.** Similar to a *covariant functor*, we can define:

Definition 2.9. A *contravariant functor* $T : \mathfrak{C} \to \mathfrak{C}'$ (between categories $\mathfrak{C}$, $\mathfrak{C}'$) is a map allocating every object X of $\mathfrak{C}$ to an object $T(X)$ of $\mathfrak{C}'$, every morphism $f : X \to Y$ in $\mathfrak{C}$ to a morphism $T(h) : T(Y) \to T(X)$ in $\mathfrak{C}'$ such that:

1. $\forall X \in \operatorname{Ob}\mathfrak{C}$, $T(1_X) = 1_{T(Y)}$.
2. For all pairs of morphisms $f : X \to Y$, $g : Y \to Z$ of $\mathfrak{C}$, $T(g \circ f) = T(f) \circ T(g)$.

The difference between a covariant and a contravariant functor is how it treats a morphism; a contravariant functor 'reverses' the order of the morphism.

Remark 2.10. From this point onwards we will assume all functors are covariant, unless specified otherwise. $\diamondsuit$

Definition 2.11. Given categories $\mathfrak{C}$, $\mathfrak{D}$ we can define the product $\mathfrak{C} \times \mathfrak{D}$ in the natural way. A functor $T : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{C}'$ is known as a *bifunctor*.

Definition 2.12. Let $T : \mathfrak{C} \to \mathfrak{C}'$ be a functor. T is *additive* if T preserves direct sums and $\forall X, Y \in \operatorname{Ob}\mathfrak{C}$, the map $T_{XY} : \operatorname{Hom}_{\mathfrak{C}}(X, Y) \to \operatorname{Hom}_{\mathfrak{C}'}(T(X), T(Y))$ given by $f \mapsto T(f)$ satisfies $T(f + g) = T(f) + T(g)$ for all $f, g \in \operatorname{Hom}_{\mathfrak{C}}(X, Y)$.

Much more is left to be said on the topic of functors, however for our purposes this is as much as we need. If the reader would like to dive a little deeper in this area, the author recommends [2, Appendix A.2] or [6, Chapter 1].

2.3. **Sequences.** Finally we introduce a core concept in homological algebra; sequences (and their exactness).

Definition 2.13. Let $\mathfrak{C}$ be an abelian category. A *sequence*

$$\dots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} X_{n+1} \longrightarrow \dots$$

is called *exact* if $\operatorname{Im} f_n = \operatorname{Ker} f_{n+1}$ for all n, where $\forall i \;\; X_i \in \operatorname{Ob}\mathfrak{C}$ and $f_i \in \operatorname{Hom}\mathfrak{C}$.
A *short exact sequence* is an exact sequence in $\mathfrak{C}$ of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

Example 2.14. Let N be a normal subgroup of a group G, with identity element 1. Then the following is an exact sequence:

$$1 \xrightarrow{i_1} N \xrightarrow{i_N} G \xrightarrow{\pi} G/N \xrightarrow{0} 1$$

This follows as:

- As i_1, i_N are the standard inclusion maps, $\operatorname{Im} i_1 = 1 = \operatorname{Ker} i_N$.
- As π is the canonical projection mapping, $\operatorname{Im} i_N = N = \operatorname{Ker} \pi$.
- By definition the zero map $0 : G/N \to 1$ sends all elements to 1, so $\operatorname{Im} \pi = \operatorname{Ker} 0$.

$\Diamond$

Definition 2.15. A short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

splits if there is a map $h : Z \to Y$ such that $g \circ h$ is the identity on Z. This is equivalent to saying $Y \cong X \oplus Z$, by the *Splitting Lemma* ([4, Theorem 1.18]).

Remark 2.16. Short sequences which split are exact. If

$$0 \longrightarrow X \xrightarrow{f} X \oplus Z \xrightarrow{\pi} Z \longrightarrow 0$$

is a sequence, we can make it exact by setting $\operatorname{Ker} \pi = X$ where π is the projection map. $\Diamond$

3. Leading to Derived Functors

As we make our way to derived functors, we first speak about projective and injective resolution. Parts of this section and the next were obtained from [2], other parts [9], and largely from [8] and [1].

Remark 3.1. We will assume all categories are abelian and all functors additive, unless stated otherwise. ♢

Definition 3.2. Let $\mathfrak{C}$, $\mathfrak{C}'$ be categories and T a functor between them. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence. We say T is

- *left exact* if
$$0 \to T(X) \xrightarrow{T(f)} T(Y) \xrightarrow{T(g)} T(Z)$$
is exact.
- *right exact* if
$$T(X) \xrightarrow{T(f)} T(Y) \xrightarrow{T(g)} T(Z) \to 0$$
is exact.
- *exact* if
$$0 \to T(X) \xrightarrow{T(f)} T(Y) \xrightarrow{T(g)} T(Z) \to 0$$
is exact.

Example 3.3. ([5, Theorem 3.1]) Let $\mathfrak{C}$ be an abelian category, and let $A \in \operatorname{Ob}\mathfrak{C}$. Define

$$F_A(X) = \operatorname{Hom}_{\mathfrak{C}}(A, X) \qquad \text{(also written } \operatorname{Hom}_{\mathfrak{C}}(A, -))$$

This defines a (covariant) left exact functor from $\mathfrak{C}$ to $\mathbf{Ab}$. We let F_A act on morphisms in $\operatorname{Hom}\mathfrak{C}$ in the natural way; by composition. Given $f : X \to Y$,

$$F_A(f) = \operatorname{Hom}_{\mathfrak{C}}(A, f) : \operatorname{Hom}_{\mathfrak{C}}(A, X) \to \operatorname{Hom}_{\mathfrak{C}}(A, Y)$$

by

$$F_A(f)(\beta) = f \circ \beta$$

♢

If we wish to construct functors repairing the failed exactness of the sequences in *Definition 3.2* there a few definitions a category must satisfy first.

Definition 3.4. An object I in a category $\mathfrak{C}$ is *injective* if given an injective morphism $f : A \to B \in \operatorname{Hom}\mathfrak{C}$ and a map $\alpha : A \to I$ there exists a map β

making the following diagram commute:

$$
\begin{array}{ccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B \\
 & & \downarrow{\scriptstyle\alpha} & \swarrow{\scriptstyle\beta} & \\
 & & I & &
\end{array}
\tag{2}
$$

One can state the idea of this in simpler terms; 'objects are highly embeddable in I'.

Definition 3.5. An object P in a category $\mathfrak{C}$ is *projective* if given a surjective morphism $g : A \to B \in \operatorname{Hom}\mathfrak{C}$ and a map $\gamma : P \to B$ there exists a map β making the following diagram commute:

$$
\begin{array}{ccccc}
 & & P & & \\
 & \swarrow{\scriptstyle\beta} & \downarrow{\scriptstyle\gamma} & & \\
A & \xrightarrow{g} & B & \longrightarrow & 0
\end{array}
\tag{3}
$$

Again one can state the idea of this in simpler terms; 'P is highly projectable".

Remark 3.6. The existence of β in *Definitions 3.4, 3.5* is known as the *universal lifting property.* ♢

Example 3.7. All free modules are projective.
Let F be a free $\mathbb{A}$-module;

$$F = \bigoplus_{j \in J} x_j \mathbb{A}$$

Using (3), the basis $\{x_j\}_{j\in J}$ gets mapped to $\{\gamma(x_j)\}_{j\in J}$ and as g is surjective, for all $j \in J$ there is an element $a_j \in A$ such that $g(a_j) = \gamma(x_j)$. Defining β such that $x_j \mapsto a_j$ means (3) commutes, making F projective, as required. ♢

Definition 3.8. A category $\mathfrak{C}$ has *enough injectives* if $\forall A \in \operatorname{Ob}\mathfrak{C}$ there is an injective morphism $A \to I$ where I is an injective object.

Definition 3.9. A category $\mathfrak{C}$ has *enough projectives* if $\forall A \in \operatorname{Ob}\mathfrak{C}$ there is a surjective morphism $P \to A$ where P is an projective object.

Definition 3.10. Let $M \in \operatorname{Ob}\mathfrak{C}$. An *injective resolution* is an exact sequence of injective objects I^j of the form:

$$0 \longrightarrow M \xrightarrow{f} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

Definition 3.11. Let $M \in \operatorname{Ob}\mathfrak{C}$. A *projective resolution* is an exact sequence of projective objects P^j of the form:

$$\cdots \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

Remark 3.12. [8, Lemma 2.2.5] If $\mathfrak{C}$ has enough injectives then every object has an injective resolution. Similarly, if $\mathfrak{C}$ has enough projectives then every object has a projective resolution. ♢

Example 3.13. Let $\mathfrak{C}$ be an abelian category. By *Example 3.3* the functor $\operatorname{Hom}_{\mathfrak{C}}(A, -)$ is a left exact functor. We can show if A is projective then $\operatorname{Hom}_{\mathfrak{C}}(A, -)$ is *exact*.
Let

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

be a short exact sequence in $\mathfrak{C}$. We wish to show

$$0 \longrightarrow \operatorname{Hom}_{\mathfrak{C}}(A, X) \xrightarrow{\ f^*\ } \operatorname{Hom}_{\mathfrak{C}}(A, Y) \xrightarrow{\ g^*\ } \operatorname{Hom}_{\mathfrak{C}}(A, Z) \xrightarrow{\ 0\ } 0$$

is an exact sequence. Assuming *Example 3.3* we just need to show $\operatorname{Im} g^* = \operatorname{Ker} 0 = \operatorname{Hom}_{\mathfrak{C}}(A, Z)$; that is, g^* is surjective. Given $\gamma \in \operatorname{Hom}_{\mathfrak{C}}(A, Z)$ the universal lifting property of A gives $\beta \in \operatorname{Hom}_{\mathfrak{C}}(A, Y)$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & A & & \\ & \overset{\beta}{\swarrow} & \downarrow{\scriptstyle \gamma} & & \\ Y & \xrightarrow{\ g\ } & Z & \longrightarrow & 0 \end{array}$$

which means $\gamma = g \circ \beta = g^*(\beta)$, i.e. g^* is surjective, as required. $\Diamond$

A more general idea of *Definitions 3.10, 3.11* is that of a chain complex. We are interested mainly in the abelian category Mod$A$, for A a K-algebra, so we will define complexes (and homologies) in this context:

Definition 3.14. A *chain complex* is a sequence

$$C_\bullet : \ldots \longrightarrow C_{n+1} \xrightarrow{\ d_{n+1}\ } C_n \xrightarrow{\ d_n\ } C_{n-1} \longrightarrow \ldots \xrightarrow{\ d_1\ } C_0 \xrightarrow{\ d_0\ } 0$$

where the C_i are right A-modules and the d_i are A-homomorphisms such that $d_n \circ d_{n+1} = 0$.

Definition 3.15. A *cochain complex* is a sequence

$$C^\bullet : 0 \xrightarrow{\ d^{-1}\ } C^0 \xrightarrow{\ d^0\ } \cdots \longrightarrow C^{n-1} \xrightarrow{\ d^{n-1}\ } C^n \xrightarrow{\ d^n\ } C^{n+1} \longrightarrow \cdots$$

where again the C^i are right A-modules and the d^i are A-homomorphisms such that $d^{n+1} \circ d^n = 0$.

As $d_n \circ d_{n+1} = 0$, $\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$. We can then define, for $n \geq 0$, the n^{th} *homology* A-module of $C_\bullet$ (and the n^{th} *cohomology* A-module of $C^\bullet$) to be

$$H_n(C_\bullet) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} \qquad \text{and} \qquad H^n(C^\bullet) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$$

respectively. This provides some sort of measure of how 'nonexact' the complex is.

Definition 3.16. Let $C_\bullet$ and $D_\bullet$ be chain complexes of right A-modules. A *chain morphism* $u : C_\bullet \to D_\bullet$ (commonly referred to as a *morphism*) is a family of A-module homomorphisms $u_n : C_n \to D_n$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle u_{n+1}} & & \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_{n-1}} & & \\ \dots & \longrightarrow & D_{n+1} & \xrightarrow{d'_{n+1}} & D_n & \xrightarrow{d'_n} & D_{n-1} & \longrightarrow & \dots \end{array} \tag{4}$$

Definition 3.17. A chain morphism $u : C_\bullet \to D_\bullet$ is known as a *quasi-isomorphism* (or by Bourbaki as a *homologism*) if the homology maps $h_n : H_n(C_\bullet) \to H_n(D_\bullet)$ are isomorphisms for all n.

Remark 3.18. If $u : C_\bullet \to D_\bullet$ is a chain morphism, the morphisms $u_n : C_n \to D_n$ induce the maps $h_n : H_n(C_\bullet) \to H_n(D_\bullet)$ due to the fact (4) commutes, i.e. $d'_n \circ u_n = u_{n-1} \circ d_n$.
For example, if $x \in \operatorname{Ker} d_n$, then

$$u_{n-1}(d_n(x)) = u_{n-1}(0) = 0 = d'_n(u_n(x)) \Rightarrow u_n(x) \in \operatorname{Ker} d'_n$$

And if $y \in \operatorname{Im} d_{n+1}$ then $\exists x$ s.t. $d_{n+1}(x) = y$ meaning

$$u_n(y) = u_n(d_{n+1}(x)) = d'_{n+1}(u_{n+1}(x)) \Rightarrow u_n(y) \in \operatorname{Im} d'_{n+1}$$

$\Diamond$

Lemma 3.19. *Let $C_\bullet$ be a chain complex of right A-modules. The following are equivalent:*

1. $C_\bullet$ *is* exact *(that is, exact at every C_n).*
2. $C_\bullet$ *is* acyclic *(that is, $H_n(C_\bullet) = 0$ for all n).*
3. *The map $0_\bullet \to C_\bullet$ is a quasi-isomorphism, where $0_\bullet$ is the chain complex of zero modules and zero maps.*

Proof. This follows from the definitions of exact, homology module and quasi-isomorphism:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$$

■

4. Chain Homotopies

Before we construct derived functors we must first build some machinery, presented in [8] and [1], that will allow us to prove the construction of derived functors is valid.

Definition 3.17 defined a quasi-isomorphism between two chain complexes, however we can expand this notion of 'similarity' between chain complexes by introducing the definition of 'similar up to a null map' - namely, *chain homotopy equivalence.*

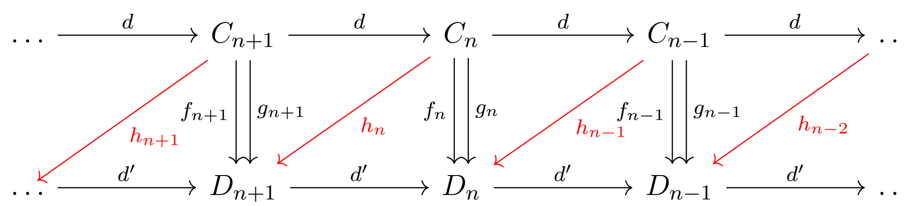
Definition 4.1. Let $\mathfrak{A}$ be an additive category, and let $C_\bullet$, $D_\bullet$ be two chain complexes. Suppose $f, g : C_\bullet \to D_\bullet$ are chain morphisms. Define a *chain homotopy* from f to g to be a collection of morphisms $h_n : C_n \to D_{n+1}$ such that

$$f_n - g_n = d_{n+1}^{D_\bullet} \circ h_n + h_{n-1} \circ d_n^{C_\bullet}$$

By setting $d = d_n^{C_\bullet}$ and $d' = d_n^{D_\bullet}$ for clarity and dropping the subscripts we can rewrite this as nicely as

$$f - g = d'h + hd$$

We can visualise these maps with the diagram below, however it should be stressed that this diagram does not commute:

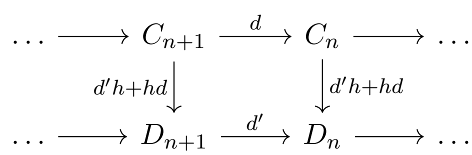


Definition 4.2. Two chain morphisms f, g are *chain homotopic* if there is a chain homotopy between them.

Remark 4.3. The map $d'h + hd$ is itself a chain morphism:

$$d'(d'h + hd) = d'd'h + d'hd = d'hd = d'hd + hdd = (d'h + hd)d$$

Namely, the following diagram commutes:



as $dd = d'd' = 0$. This map is called *null homotopic*. Then *Definition 4.2* can be restated as "f, g are chain homotopic if $f - g$ is null homotopic". $\diamondsuit$

Finally we define:

Definition 4.4. A chain morphism $f : C_\bullet \to D_\bullet$ is a *chain homotopy equivalence* (known by Bourbaki as a *homotopism*) if there is a map $g : D_\bullet \to C_\bullet$ such that $f \circ g$ and $g \circ f$ are chain homotopic to the identity maps of $D_\bullet$ and $C_\bullet$ respectively.
The complexes $C_\bullet$, $D_\bullet$ are said to be *homotopy equivalent* if there is a chain homotopy equivalence between them.

As a demonstration of the similarity of chain homotopic morphisms we have the following lemma:

Lemma 4.5. ([1, Chapter IX, §4]) *If $f, g : C_\bullet \to D_\bullet$ are chain homotopic morphisms then f and g induce the same homology maps $H_n(C_\bullet) \to H_n(D_\bullet)$ for all n.*

Proof. We will prove this statement element-wise, by taking an element $\mathfrak{a} \in H_n(C_\bullet)$ and mapping it into $H_n(D_\bullet)$, for a fixed n.

Suppose $\mathfrak{a}$ as an equivalence class is represented by $a \in \operatorname{Ker} d_n^{C_\bullet}$, and its images in $H_n(D_\bullet)$ are given by $f_n(a)$, $g_n(a)$ from the morphisms f and g. As f and g are homotopic, there are morphisms h_n such that

$$f_n(a) - g_n(a) = d_{n+1}^{D_\bullet}(h_n(a)) + h_{n-1}(d_n^{C_\bullet}(a))$$

Since $a \in \operatorname{Ker} d_n^{C_\bullet}$ the latter term vanishes, so we conclude $f_n(a) - g_n(a) \in \operatorname{Im} d_{n+1}^{D_\bullet}$. As in $H_n(D_\bullet)$ we take elements modulo $\operatorname{Im} d_{n+1}^{D_\bullet}$, $f_n(a) - g_n(a) = 0$. We conclude f and g induce the same homology maps, as required. ∎

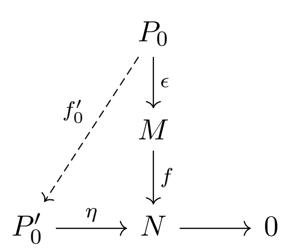
Another idea that will be useful in §5 is that a morphism of objects can be extended to a chain morphism of the projective resolutions of these objects which is unique up to chain homotopy:

Lemma 4.6. *Suppose $M, N \in \operatorname{Ob}\mathfrak{A}$ and f is a morphism between them. If $\epsilon : P_\bullet \to M$ is a projective resolution of M, and $\eta : P'_\bullet \to N$ a projective resolution of N then there is a chain morphism $f' : P_\bullet \to P'_\bullet$ 'lifting' f, in the sense $f \circ \epsilon = \eta \circ f'_0$.*
Moreover this chain morphism is unique up to chain homotopy, that is, if g' is another chain morphism satisfying the same conditions as f', then f' and g' are chain homotopic.

Proof. (Adapted from [7, Proposition 3.3.1])
We wish to first prove the existence of the f'_i in the following diagram:

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & \dots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f'_{n+1}} & & \Big\downarrow{\scriptstyle f'_n} & & & & \Big\downarrow{\scriptstyle f'_1} & & \Big\downarrow{\scriptstyle f'_0} & & \Big\downarrow{\scriptstyle f} & & \\
\dots & \longrightarrow & P'_{n+1} & \xrightarrow{d'_{n+1}} & P'_n & \xrightarrow{d'_n} & \dots & \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\eta} & N & \longrightarrow & 0
\end{array}$$

We do so by induction, and it nearly immediately follows from the universal lifting property.
For $n = 0$, by the universal lifting property f'_0 exists, making the following diagram commute:



We now assume $f'_0, \dots, f'_n$ have been defined. Note that, by exactness,

$$x \in \operatorname{Im} d_{n+1} \Rightarrow d_n(x) = 0 \Rightarrow d'_n f'_n(x) = f'_{n-1} d_n(x) = 0$$

Here we also use the fact

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & P_{n-1} \\ \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f'_{n-1}} \\ P'_n & \xrightarrow{d'_n} & P'_{n-1} \end{array}$$

is (inductively) commutative. We can conclude $f'_n(\operatorname{Im} d_{n+1}) \subseteq \operatorname{Ker} d'_n = \operatorname{Im} d'_{n+1}$ and again by the universal lifting property f'_{n+1} exists, making the following diagram commute:

$$\begin{array}{ccccc} & & P_{n+1} & & \\ & \swarrow{\scriptstyle f'_{n+1}} & \downarrow{\scriptstyle d_{n+1}} & & \\ & & \operatorname{Im} d_{n+1} & & \\ & & \downarrow{\scriptstyle f'_n} & & \\ P'_{n+1} & \xrightarrow{d'_{n+1}} & \operatorname{Im} d'_{n+1} & \longrightarrow & 0 \end{array}$$

This finishes the induction. We now need to prove f' is unique up to chain homotopy:

Suppose there is another morphism $g' : P_\bullet \to P'_\bullet$ satisfying the conditions of the lemma. We will construct the maps h_n to make f' and g' homotopic, and again do so by induction:

Note $\eta \circ f'_0 = f \circ \epsilon = \eta \circ g'_0$ meaning $\eta \circ (f'_0 - g'_0) = 0$. So the map $(f'_0 - g'_0)$ takes elements into $\operatorname{Ker} \eta = \operatorname{Im} d'_1$. By the universal lifting property there exists a map h_0 making the following diagram commute:

$$\begin{array}{ccccc} & & P_0 & & \\ & \swarrow{\scriptstyle h_0} & \downarrow{\scriptstyle f'_0 - g'_0} & & \\ P'_1 & \xrightarrow{d'_1} & \operatorname{Im} d'_1 & \longrightarrow & 0 \end{array}$$

This satisfies the base case of the induction. Now assume $h_0, \dots, h_n$ have been constructed. Note by our induction we can also assume

$$f'_n - g'_n = d'_{n+1} \circ h_n + h_{n-1} \circ d_n$$

In the same style as before, we will want $f'_{n+1} - g'_{n+1} - h_n \circ d_{n+1}$ to take values in $\operatorname{Ker} d_{n+1} = \operatorname{Im} d_{n+2}$. This follows as:

$$
\begin{aligned}
&d'_{n+1} \circ (f'_{n+1} - g'_{n+1} - h_n \circ d_{n+1}) \\
&= d'_{n+1} f'_{n+1} - d'_{n+1} g'_{n+1} - d'_{n+1} h_n d_{n+1} && \text{(Dropping the $\circ$ for clarity)} \\
&= f'_n d_{n+1} - g'_n d_{n+1} - d'_{n+1} h_n d_{n+1} && \text{(Using commutativity of f' and g')} \\
&= (f'_n - g'_n - d'_{n+1} h_n) d_{n+1} \\
&= (h_{n-1} d_n) d_{n+1} && \text{(By induction)} \\
&= 0 && \text{(As $d_n d_{n+1} = 0$)}
\end{aligned}
$$

Hence h_{n+1} follows as before by the universal lifting property making the following diagram commute:

$$
\begin{array}{ccccc}
 & & P_{n+1} & & \\
 & \swarrow^{h_{n+1}} & \downarrow{\scriptstyle f'_{n+1} - g'_{n+1} - h_n d_{n+1}} & & \\
P'_{n+2} & \xrightarrow{d'_{n+2}} & \operatorname{Im} d'_{n+2} & \longrightarrow & 0
\end{array}
$$

Therefore by construction f' and g' are chain homotopic, so f' is unique up to chain homotopy, as required. ■

Remark 4.7. Anything proven about chain morphisms, projective resolutions or homologies in this section can also be proven about cochain morphisms, injective resolutions and cohomologies. In particular, *Lemma 4.5* and *Lemma 4.6* will in §5 use cohomologies and injective resolutions respectively. ♢

5. Derived Functors

We now turn to the construction of *right derived functors for left exact sequences* using the tools we have built in the previous sections.

Let $\mathfrak{C}$ be a category with enough injectives, and F a left exact functor. Begin with $M \in \operatorname{Ob}\mathfrak{C}$. As $\mathfrak{C}$ has enough injectives, there is an injective resolution of M

$$0 \longrightarrow M \xrightarrow{f} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

which is a long exact sequence. We rewrite this as two cochain complexes

$$
\begin{array}{lccccccccc}
M^\bullet : & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow{\scriptstyle f} & & & & & & \\
I^\bullet : & 0 & \xrightarrow{d^{-1}} & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \longrightarrow & \cdots
\end{array}
$$

where the two complexes are 'similar' in the sense that their cohomologies at (almost) every point agree. We will work with the $I^\bullet$ cochain complex (in essence suppressing the M object), and we keep in mind it might not be

fully exact.
Applying the left exact functor F we obtain

$$F(I^\bullet): 0 \xrightarrow{F(d^{-1})} F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^2) \longrightarrow \cdots$$

We then compute its cohomology at the i^{th} spot and call the resulting object $R^iF(M)$:

$$R^iF(M) := H^i(F(I^\bullet)) = \operatorname{Ker} F(d^i)/\operatorname{Im} F(d^{i-1})$$

and note in particular as F is left exact we have the exact sequence

$$0 \longrightarrow F(M) \xrightarrow{F(f)} F(I^0) \xrightarrow{F(d^0)} F(I^1)$$

and as $f: M \to I^0$ is an injection we conclude

$$R^0F(M) = \operatorname{Ker} F(d^0)/\operatorname{Im} F(d^{-1}) = \operatorname{Im} F(f)/0 \;\cong\; F(M)$$

We say F has been *derived* to form R^iF, namely R^iF is a *derived functor*.

Remark 5.1. In a similar way we can construct the *left derived functors for right exact sequences* by:

(1) Assuming $\mathfrak{C}$ has enough projectives.
(2) Taking a projective resolution of $N \in \operatorname{Ob}\mathfrak{C}$, and forming the chain complex $K_\bullet$.
(3) Computing the i^{th} homology and defining

$$L_iF(N) = H_i(F(K_\bullet)) = \operatorname{Ker} F(d_i)/\operatorname{Im} F(d_{i+1})$$

◊

Remark 5.2. There is a correspondence between right and left derived functors:

$$R^iF(M) = (L_iF^{op})^{op}(M)$$

which follows from the correspondence between injective resolutions in $\mathfrak{C}$ and projective resolutions in $\mathfrak{C}^{op}$. ◊

We now need to prove that the constructed functor 'fixes' exactness. First, two lemmas which ensure the derived functors are well defined:

Lemma 5.3. *$R^iF(M)$ does not depend on the injective resolution of M. That is, if $J^\bullet$ is a second resolution of M, then*

$$H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$$

Proof. By *Lemma 4.6* there is a cochain morphism $f: I^\bullet \to J^\bullet$ lifting the identity map on M, $\mathrm{id}_{\mathfrak{C}}$, that is unique up to cochain homotopy. However by *Lemma 4.5* all homotopic cochain morphisms induce the same cohomology map $f^*: H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$, so this map is canonical. Similarly there is a cochain morphism $g: J^\bullet \to I^\bullet$ lifting $\mathrm{id}_{\mathfrak{C}}$ and a corresponding canonical map g^*. Since $g \circ f$ and $\mathrm{id}_{I^\bullet}$ are cochain morphisms of $I^\bullet \to I^\bullet$ lifting $\mathrm{id}_{\mathfrak{C}}$, they are homotopic and we conclude

$$g^* \circ f^* = (g \circ f)^* = (id_{I^\bullet})^* = \text{ identity on } H^i(F(I^\bullet))$$

Similarly $f^* \circ g^* = (id_{J^\bullet})^*$. Thus f and g are quasi-isomorphisms and

$$H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$$

as required. ■

Lemma 5.4. *R^iF is a functor. In particular, a morphism $f : M \to N$ yields a morphism $R^iF(f) : R^iF(M) \to R^iF(N)$.*

Sketch proof. If $I^\bullet_M$, $I^\bullet_N$ are injective resolutions of M and N respectively, there is a cochain morphism $f' : I^\bullet_M \to I^\bullet_N$ lifting f unique up to cochain homotopy. Thus by the same reasoning as *Lemma 5.3* the map

$$f'_* : H^i(F(I^\bullet_M)) \to H^i(F(I^\bullet_N))$$

is canonical. This will be the map we want; $R^iF(f) = f'_*$.
The other properties of a covariant functor (preservation of the identity and composition) follow naturally in suit. ■

We now wish to show right derived functors turn the sequences formed by left exact functors on short exact sequences into long exact sequences.

Theorem 5.5. *Suppose $\mathfrak{C}$ and F are as previously given. Given a short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{5}$$

the following is a long exact sequence:

$$\begin{array}{l} 0 \longrightarrow F(X) \xrightarrow{f^0} F(Y) \xrightarrow{g^0} F(Z) \xrightarrow{h^0} \\ \hookrightarrow R^1F(X) \xrightarrow{f^1} R^1F(Y) \xrightarrow{g^1} R^1F(Z) \xrightarrow{h^1} \\ \hookrightarrow R^2F(X) \xrightarrow{f^2} R^2F(Y) \xrightarrow{g^2} \cdots \end{array} \tag{6}$$

Proof. This theorem is proven in the context of left derived functors and projective resolutions, however by *Remark 5.2* we can convert between left and right derived functors and projective and injective resolutions. Thus, we will prove the corresponding result for left derived functors instead. We need to use two lemmas to prove this statement:

Lemma 5.6. ***Horseshoe Lemma.*** *Given (5) in $\mathfrak{C}$ with $P^\bullet_X, P^\bullet_Z$ the projective resolutions of X, Z respectively, there exists a projective resolution $P^\bullet_Y$ of Y such that*

$$0 \longrightarrow P^\bullet_X \longrightarrow P^\bullet_Y \longrightarrow P^\bullet_Z \longrightarrow 0 \tag{7}$$

is an exact sequence.

Sketch proof. Set $P_Y^i = P_X^i \oplus P_Z^i$. This will form a projective resolution of Y, and all rows of the following diagram will be exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_X^0 & \longrightarrow & P_X^0 \oplus P_Z^0 & \longrightarrow & P_Z^0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_X^1 & \longrightarrow & P_X^1 \oplus P_Z^1 & \longrightarrow & P_Z^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_X^2 & \longrightarrow & P_X^2 \oplus P_Z^2 & \longrightarrow & P_Z^2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{8}
$$

■

Lemma 5.7. *Given (7) and F an additive functor of abelian categories, then*

$$0 \longrightarrow F(P_X^\bullet) \longrightarrow F(P_Y^\bullet) \longrightarrow F(P_Z^\bullet) \longrightarrow 0 \tag{9}$$

is an exact sequence.

Proof. Since P_Z^i is projective,

$$
\begin{array}{ccccccccc}
 & & & & & & P_Z^i & & \\
 & & & & & \overset{\beta}{\swarrow} & \downarrow \text{id} & & \\
0 & \longrightarrow & P_X^i & \xrightarrow{f} & P_Y^i & \xrightarrow{g} & P_Z^i & \longrightarrow & 0
\end{array} \tag{10}
$$

there is a map $\beta : P_Z^i \to P_Y^i$ that composes with g to form the identity id - i.e. $\beta = g^{-1}$. Thus P_Y^i can be written as $P_X^i \oplus P_Z^i$, meaning the sequence (10) splits. Then

$$0 \longrightarrow F(P_X^i) \longrightarrow F(P_Y^i) \longrightarrow F(P_Z^i) \longrightarrow 0$$

splits, as F is additive. By *Remark 2.16*, this sequence is exact thus (9) is exact, as required. ■

Returning to the proof of Theorem 5.5*:*
Given (5), we obtain (7) by taking the correct projective resolutions. This is exact by *Lemma 5.6* and by *Lemma 5.7*, (9) is exact. Expanding (9) we obtain:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_X^0) & \longrightarrow & F(P_Y^0) & \longrightarrow & F(P_Z^0) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_X^1) & \longrightarrow & F(P_Y^1) & \longrightarrow & F(P_Z^1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_X^2) & \longrightarrow & F(P_Y^2) & \longrightarrow & F(P_Z^2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{11}
$$

We note that the homology objects of the column complexes of (11) are exactly the left derived functors of F, as we defined them.

Finally, by a diagram chase ([1, Chapter IX, Theorem 3.5]) of these homology objects, we obtain the long exact sequence

$$
\begin{array}{l}
\longrightarrow F(X) \xrightarrow{f^0} F(Y) \xrightarrow{g^0} F(Z) \longrightarrow 0 \\
\xrightarrow{h^1} L_1F(X) \xrightarrow{f^1} L_1F(Y) \xrightarrow{g^1} L_1F(Z) \\
\xrightarrow{h^2} \\
\cdots \longrightarrow L_2F(X) \xrightarrow{f^2} L_2F(Y) \xrightarrow{g^2} L_2F(Z)
\end{array}
$$

using the *Snake Lemma*. Thus, based on these results, we can conclude (6) is a long exact exact sequence too, as required. ∎

Lemma 5.8. *If A is injective then $R^iF(A) = 0$ for $i \geq 1$.*

Proof. If A is injective, it has an injective resolution

$$X^\bullet : 0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

so

$$R^iF(A) = H^i(F(X^\bullet)) = 0$$

for $i \geq 1$, by induction on the length of $X^\bullet$. ∎

Corollary 5.9. *If A is projective then $L_iF(A) = 0$ for $i \geq 1$.* ∎

Remark 5.10. The famous *Snake Lemma* can be viewed as a 'special case' of *Theorem 5.5* - it should be noted this is *not* a proof of the Snake Lemma, rather a way to rewrite it to demonstrate the previous theorem. Recall the Snake Lemma says for a commutative diagram of exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L' & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & \longrightarrow & 0 \end{array} \tag{12}$$

there is a connecting homomorphism δ

$$\begin{array}{ccccccccc} & & \mathrm{Ker} f & \xrightarrow{u} & \mathrm{Ker} g & \xrightarrow{v} & \mathrm{Ker} h & \overset{\delta}{\dashrightarrow} & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L' & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & \dashrightarrow & \mathrm{Coker} f & \xrightarrow{u'} & \mathrm{Coker} g & \xrightarrow{v'} & \mathrm{Coker} h & & \end{array} \tag{13}$$

making the sequence

$$0 \longrightarrow \mathrm{Ker} f \xrightarrow{u} \mathrm{Ker} g \xrightarrow{v} \mathrm{Ker} h \xrightarrow{\delta} \mathrm{Coker} f \xrightarrow{u'} \mathrm{Coker} g \xrightarrow{v'} \mathrm{Coker} h \longrightarrow 0$$

exact. If we view the columns of (12) as complexes

$$L^\bullet : \ 0 \longrightarrow L \xrightarrow{f} L' \longrightarrow 0 \longrightarrow \cdots$$

$$M^\bullet : 0 \longrightarrow M \xrightarrow{g} M' \longrightarrow 0 \longrightarrow \cdots$$

$$N^\bullet : \ 0 \longrightarrow N \xrightarrow{h} N' \longrightarrow 0 \longrightarrow \cdots$$

then (12) is the expansion of the short exact sequence

$$0 \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet \longrightarrow 0$$

of these complexes. The Snake Lemma then tells us there is an exact sequence

$$0 \longrightarrow H^0(L^\bullet) \longrightarrow H^0(M^\bullet) \longrightarrow H^0(N^\bullet) \xrightarrow{\delta} H^1(L^\bullet) \longrightarrow H^1(M^\bullet) \longrightarrow H^1(N^\bullet) \longrightarrow 0$$

which we can see is a special case of (6). ◊

5.1. **Applications: the Ext functor.** A popular example of derived functors are the Ext functors obtained by deriving the Hom functors. We have already come across Hom functors before; in *Example 3.3* we defined the Hom functor on abelian categories:

$$\mathrm{Hom}_{\mathfrak{C}}(A,-) : \mathfrak{C} \to \mathbf{Ab} \qquad X \mapsto \mathrm{Hom}_{\mathfrak{C}}(A,X)$$

and found it was a left exact functor. Using the results of §5 we obtain the i^{th} right derived functor of $\mathrm{Hom}_{\mathfrak{C}}(A,-)$ as

$$\mathrm{Ext}^i_{\mathfrak{C}}(A,-) := R^i \mathrm{Hom}_{\mathfrak{C}}(A,-)$$

If A is a K-algebra we can similarly define the functor $\mathrm{Hom}_K(M,-)$ over $\mathrm{Mod}A$ and obtain the n^{th} derived functor

$$\mathrm{Ext}^n_K(M,-) : \mathrm{Mod}A \to \mathrm{Mod}K$$

of $\mathrm{Hom}_K(M,-)$ (for a fixed A-module M).

There are two functors associated with Hom - the covariant $\mathrm{Hom}_K(M,-)$ and the contravariant $\mathrm{Hom}_K(-,M)$. This leads to covariant and contravariant n^{th} right derived functors $\mathrm{Ext}^n_K(M,-)$ and $\mathrm{Ext}^n_K(-,M)$, obtained from taking injective and projective resolutions respectively[2]. We might expect these two derived functors to be very different in behaviour, however the bifunctor $\mathrm{Ext}^n_K(-,-)$ (which is contravariant in the first variable and covariant in the second) surprisingly works as a derived functor for both the covariant and contravariant Hom functors[3].

Example 5.11. We will compute $\mathrm{Ext}^n_{\mathbb{Z}}(-,\mathbb{Z})(\mathbb{Z}_p) = \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})$.

First, take a projective resolution of $\mathbb{Z}_p$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\equiv_p} \mathbb{Z}_p \longrightarrow 0$$

Applying the right exact contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$, we get the exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{(\times p)^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{(\equiv_p)^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})$$

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z}) = 0$ we get the following:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(\times p)^*} \mathbb{Z} \xrightarrow{(\equiv_p)^*} 0$$

Finally,

- $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z}) \cong 0$
- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z}) = \mathrm{Ker}(\times p)^*/\mathrm{Im}(\equiv_p)^* = \mathbb{Z}_p/0 \cong \mathbb{Z}_p$
- $\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z}) = 0$ for $i \geq 2$.

[2]Applying a contravariant functor to a projective resolution also leads to a right derived functor.

[3] This correspondence is explored in more detail in [1, Chapter VIII, §6.4].

Note that by *Corollary 5.9* this means $\mathbb{Z}_p$ isn't projective as a $\mathbb{Z}$-module. $\Diamond$

Remark 5.12. If G is a finitely generated abelian group, then $\operatorname{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z}) \cong G$. This follows from the *Fundamental Theorem of Finitely Generated Abelian Groups*, which states G can be written as a direct sum involving $\mathbb{Z}^n$ and $\mathbb{Z}_p$. $\Diamond$

Remark 5.13. By *Lemma 5.8* if N is an injective module we can immediately conclude

$$\forall M,\ \operatorname{Ext}^n_K(M,-)(N) = \operatorname{Ext}^n_K(M,N) = 0$$

for all $n \geq 1$. $\Diamond$

6. Closing remarks

Definition 6.1. A (covariant) *cohomological δ-functor* between abelian categories $\mathfrak{A}$ and $\mathfrak{B}$ is a collection of additive functors $T^n : \mathfrak{A} \to \mathfrak{B}$ (indexed by nonnegative integers) together with a family of morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ for each short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{14}$$

in $\mathfrak{A}$, such that the following two properties hold:

(1) For each short exact sequence (14) there is a long exact sequence

$$0 \longrightarrow T^0(A) \longrightarrow T^0(B) \longrightarrow T^0(C) \xrightarrow{\delta^0} T^1(A) \longrightarrow \cdots \longrightarrow T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \longrightarrow T^{n+1}(B) \longrightarrow \cdots$$

(2) Each morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \downarrow & & \downarrow \\ T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A') \end{array}$$

(This property is known as *naturality*.)

This definition generalises the notion of right derived functors; by *Theorem 5.5* they satisfy (1) and their naturality is proven in [8, Theorem 2.4.6][4]. Similarly the concept of a *homological* δ-functor can be defined to generalise left derived functors.

Introduced by Grothendieck [3] in his famous *Tôhoku paper*, this context is intended to be the appropriate setting in which to treat and further the development of derived functors.

We have thus plotted the course of derived functors from their roots in categories and exact sequences to the point where we see they are part of a much larger, more general framework, and it is at this point we leave the reader to explore [8, Chapter 2] and [3] for more information on the subject.

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