HIGHER CATEGORY THEORY

BRIAN TYRRELL

ABSTRACT. In this essay we introduce higher category theory through one definition of 'n-category', using opetopic sets. We then briefly showcase some applications of higher category theory to topology and physics (amongst other subjects) and conclude by conjecturing a connection to natural language processing.

1. Introduction

There is a well-worn joke that is appropriate for our setting:

> *x_1^3 and x_2^7 are sitting in a bar, chatting to pass the time. Eventually, the question of religion arises.* Are you religious? *asks x_1^3.* Well, *x_2^7 replies,* I do believe in higher powers.

To the author it seems *belief* is already a fundamental component of higher category theory: so much is unknown yet so much is expected to be discovered in the coming decades. Crucially, as its stands there are several approaches available to us in defining what an *n-category* is, and we are unsure if the definitions we produce are all equivalent! Cheng & Lauda [CL04] liken higher categories to conquering a mountain and say "intrepid explorers have made the ascent, each taking a different route" but most importantly we are undecided if "we are even climbing the same mountain". Moving in this direction, Baez [Bae97] describes "preliminary chores" which first must be completed before applications to geometry, homotopy theory, type theory and even physics (such as topological quantum field theory, or more generally n-categorical physics [BL09]) can be fully realised. Such applications the author will mention briefly in §3 and we will reference these 'preliminary chores' again at the start of §4. Before this we will spend some time introducing Baez & Dolan's definition of a *weak n-category* [BD98] in §2, though the reader should be forewarned that (for the sake of understanding) some corners will be cut; full details are available in [BD98], our definition will be plucked from [Bae97], and a straightforward presentation without technicalities can be found in [CL04, Chapt. 4]. The reader is also encouraged to 'test their faith' and explore other chapters of [CL04] (in particular, [CL04, Appendix A.1]) and Leinster's survey [Lei02] for alternate definitions of *n-category.*

2. What is an n-Category?

In some sense, this is the £1,000,000 question. Though the basic intuition and idea is clear amongst authors listed in [CL04] and [Lei02], approaches to formalise this concept can take radically different paths. For us, we will follow the programme outlined in [Bae97][1].

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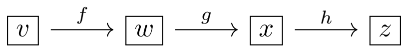
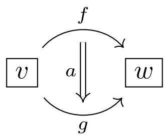
[1] The author is no set theorist, so concerns about the 'sizes' of sets are largely swept under the rug and dealt with elsewhere (cf. [Zhe14, ML78]).

Very simply, we can recursively build the definition of *n-category* as follows: a 0-*category* is a set. A 1-*category* is a category; a collection of 'objects' and 'morphisms' between those objects, obeying composition laws. A 2-*category* is a collection of 'objects', 'morphisms' between those objects, and '2-morphisms' between those morphisms, all obeying "reasonable" composition laws - which we will specify later. This process continues, all the way up to n-morphisms for n-categories. The crux of the matter is the notion of "sameness" in each of these types of category. In a set (a 0-category) objects are either the same (they are equal) or they are not. Life is simple. In a category (a 1-category) objects can either be equal or isomorphic. What about morphisms? In a category they can either be the same map, or not. There is no notion of 'isomorphic' morphisms - for this level of detail, we must turn to higher categories!

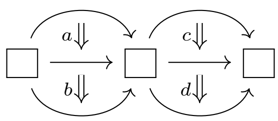
Consider the category **Cat** whose objects are categories and whose morphisms are functors. Here we do have a concept of 'isomorphic morphisms'; that of a natural transformation. Let us represent objects in this category by boxes and morphisms as arrows between these points:



Cat obeys the usual composition law of $h \circ (g \circ f) = (h \circ g) \circ f$. Between morphisms in this category we can have a natural transformation (2-morphism), represented as follows:



There are two ways to compose 2-morphisms corresponding to our geometrical intuition; vertically and horizontally.



Cat also obeys the following natural composition law:

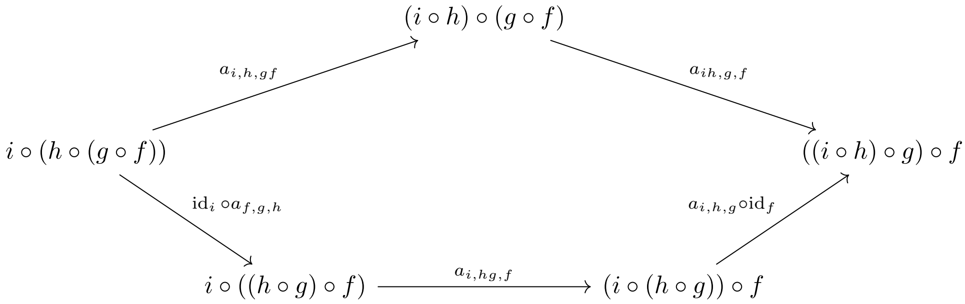
$$(d \circ_v c) \circ_h (b \circ_v a) = (d \circ_h b) \circ_v (c \circ_h a), \tag{1}$$

where "$\circ_v$" denotes vertical composition, and "$\circ_h$" denotes horizontal composition. This makes **Cat** a *strict 2-category:*

Definition 2.1. A *strict 2-category* consists of a collection of *objects*, for every two objects a collection of *morphisms* between them, and for every two morphisms a collection of *2-morphisms* between them, such that the objects and morphisms satisfy the usual rules holding in a category and the 2-morphisms satisfy the composition rule (1) in addition to the expected axioms of associativity and identity.

We can weaken this definition to be a bit more useful by instead requiring the equations of associativity, identity for *morphisms* to hold only up to natural isomorphism:

Definition 2.2. A *weak 2-category* or *bicategory* consists of a collection of *objects*, for every two objects a collection of *morphisms* between them, and for every two morphisms a collection of *2-morphisms* between them, such that the 2-morphisms satisfy the composition rule (1) in addition to the expected axioms of associativity and identity, and for each triple of morphisms f, g, h, instead of an associative law there exists an invertible associator 2-morphism $a_{h,g,f} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$ such that the following diagram commutes:



In addition there are invertible left & right identity morphisms $l_f : f \circ \mathrm{id}_x \Rightarrow f$, $r_f : \mathrm{id}_y \circ f \Rightarrow f$ satisfying a diagram of their own (cf. [Bae97, §2]).

Remark 2.3. In *Definition 2.2* we still require the 2-morphisms to satisfy a 'strict' law, such as the equation (1) or the equations for associativity and identity. If we wished to 'weaken' this structure and accept 2-morphisms that satisfy, say, associativity "up to isomorphism" we would need to turn to 3-categories; there the notion of 'isomorphism' for 2-morphisms exists, as it does not in the original 2-categorical structure. □

As [Bae97] remarks, the definition of bicategory seems initially to be more "clumsy" than that of a strict 2-category, however it is (arguably) a more useful notion as in many applications typically "everything is true up to something" which is reflected more in the second type of structure. As Baez notes, "...the whole point of introducing $(n+1)$-morphisms is to allow n-morphisms to be isomorphic rather than merely equal. From this point of view, it was inappropriate to have imposed equational laws between 1-morphisms in the definition of a strict 2-category, and the definition of a bicategory corrects this problem".

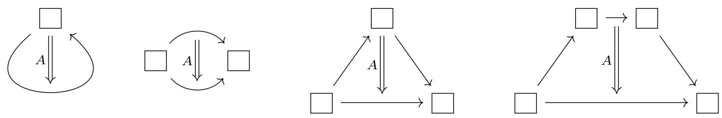
Example 2.4. For an example of a strict (weak) 3-category one can consider **2Cat**, the category with objects that are strict (weak) 2-categories, whose morphisms are (somewhat) structure-preserving (pseudo)functors, whose 2-morphisms are, as expected, (pseudo)natural transformations and whose 3-morphisms are known as (pseudo)*modifications*.

In the case of a strict 3-category, these satisfy naturality conditions, composition conditions, associativity laws and identity laws. (In the case of a weak 3-category, pseudofunctors preserve structure only up to 2-isomorphism, which in turn most satisfy some "coherence" laws involving 3-morphisms, e.g. see the above pentagon diagram. In a 3-category only the 3-morphisms satisfy equational laws, as there is no room for weakening.)

The reader is invited to consult [Bae97, §3.1] for some diagrams which possibly clarify the role of natural transformations and modifications. □

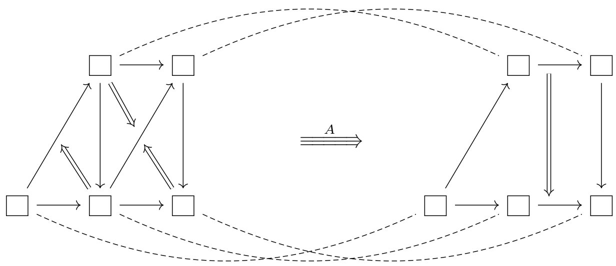
2.1. **Opetopes.** We will now present Baez & Dolan's definition of a weak n-category via *opetopic sets*. We will begin with the notion of an *opetope* (a portmanteau of "operation" and "polytope"): the category of opetopes has a somewhat involved definition which can be found in [Bae97] and [BD98] so we will instead present a brief description of what exactly an opetope looks like, more in the style of [CL04]. The motivation for opetopes is that the basic shapes of j-morphisms correspond to the j-dimensional opetopes, and the permitted styles of gluing together j-dimensional opetopes (i.e. composing j-morphisms) and the coherence laws satisfied correspond to higher dimensional opetopes still.

Definition 2.5. The only 0-dimensional opetope is the box $\square$. The only 1-dimensional opetope is the arrow $\square \xrightarrow{A} \square$. There are many ways to glue together 1-dimensional opetopes:



etc.

These form the 2-dimensional opetopes. An example of a 3-dimensional opetope is the following:



This is a three dimensional shape whose front are the left opetopes and whose rear is the right opetope; the dotted lines denote that the indicated boxes and edges should be glued together (picture the right opetope as a flat base and the left opetopes as curved faces above the base, with the triple arrow $\Rrightarrow$ between the two faces).

Each of these shapes is known as a *cell*. In all of these examples the 'base' (what the arrow with the most lines is pointing to) is known as the *outface* and all other faces of the shape are *infaces*. In general, a $(n+1)$-opetope has any number of infaces and one outface.

Definition 2.6. An *opetopic set* is a presheaf on the category **Op** of opetopes[2].

Very simply, a weak n-category is an opetopic set satisfying two important properties, which we shall cover next.

2.2. **Niches.** Let us represent a j-dimensional cell A in an opetopic set as

$$(a_1, \ldots, a_k) \xrightarrow{A} a',$$

where $a_1, \ldots, a_k$ are the infaces and a' is the outface. We will write this as $(A; a_1, \ldots, a_k; b)$.

Definition 2.7. A *frame* is a cell $(?; a_1, \ldots, a_k; a')$ with the data of A missing. A *niche* is a frame $(?; a_1, \ldots, a_k; ?)$ with the outface missing. A *punctured niche* is a niche $(?; a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k; ?)$ with one inface missing.

To the author these are formal constructions; the important definition is the next:

Definition 2.8. If a frame, niche, or punctured niche can be extended to a cell, the cell is called an *occupant* of the frame/niche/punctured niche.

In addition, occupants of the same niche are called *niche competitors*.

[2]Whose objects are j-dimensional opetopes for all $j \geq 0$ and whose morphisms describe how one opetope is included as a face of another opetope.

2.3. **Universality.** As [Bae97] remarks, the "main role" of universality (in this context) is in the definition of a *composite* outface, which we shall give now:

Definition 2.9. Given a universal occupant U of a j-dimensional niche

$$(a_1, \ldots, a_k) \xrightarrow{U} b,$$

we call b a *composite* of $a_1, \ldots, a_k$.

The idea of universality is one we are already familiar with from a standard category-theoretic context:

Definition 2.10. (Sketch) A j-dimensional niche occupant U is *universal* if all of its niche competitors factor through it, "up to equivalence".

This definition is a sketch as "up to equivalence" is a tricky thing to capture and explain to full satisfaction: the reader is welcome to skip *Definition 2.11 & Remark 2.12* to avoid technicalities.

To start with, we will fix an $n \in \mathbb{N}$ and define *universal* (and *balanced*) relative to an n-category. The definitions of these two types of niche structures are given inductively in an interdependent way however care has been taken so the definitions are not circular.

Definition 2.11.

(1) A j-dimensional niche occupant $(U; a_1, \ldots, a_k; b)$ is *universal* if and only if either $j > n$ and U is the only occupant of its niche, or $j \leq n$ and for any frame competitor b' of b the $(j+1)$-dimensional punctured niche

$$\big(?; (U; a_1, \ldots, a_k; b), (?; b; b'); (?; a_1, \ldots, a_k; b')\big)$$

and its mirror image

$$\big(?; (?; b; b'), (U; a_1, \ldots, a_k; b); (?; a_1, \ldots, a_k; b')\big)$$

are *balanced*.

(2) A l-dimensional punctured niche $(?; a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k; ?)$ is *balanced* if any only if either $l > n + 1$ or
- Any outface extension $(?; a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k; b)$ extends further to $(U; a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k; b)$ with U *universal*, and
- For any *universal* occupant $(U; a_1, \ldots, a_k; b)$ and frame competitor a_i' of a_i the $(l+1)$-dimensional punctured niche

$$\big(?; (?; a_i'; a_i), (U; a_1, \ldots a_k; b); (?; a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_k; b)\big)$$

  and its mirror image

$$\big(?; (U; a_1, \ldots, a_k; b), (?; a_i'; a_i); (?; a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_k; b)\big)$$

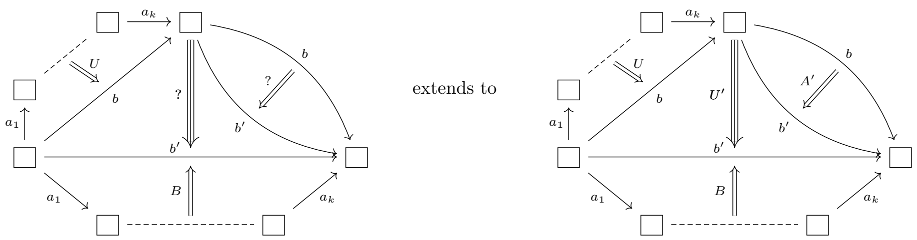
  are *balanced*.

Remark 2.12. A n-dimensional niche occupant $(U; a_1, \ldots, a_k; b)$ is universal if and only if for any frame competitor b' of b, any outface extension

$$\big(?; (U; a_1, \ldots, a_k; b), (?; b; b'); (B; a_1, \ldots, a_k; b')\big)$$

extends to a unique occupant $\big(U'; (U; a_1, \ldots, a_k; b), (A'; b; b'); (B; a_1, \ldots, a_k; b')\big)$ (and the same for the mirror image). Pictographically[3],

[3] In reality this picture should be three dimensional with the outface a 'base' as described in *Definition 2.5* (or [Bae97, p. 19]), but the author has suggestively drawn this as a two dimensional figure.



where the occupant to the right is unique, and the same diagram can be drawn for the mirror of the left opetope. Colloquially this expresses "n-dimensional niches are universal if and only if the composite of the infaces *equals* the outface". □

We are finally ready for Baez & Dolan's definition of a weak n-category. Recall *Definition 2.9* of a *composite*; in light of the definition of universality we see that an outface is really only a composite if and only if it is literally a composite of the infaces "up to equivalence" (cf. [BD98, Prop. 55] too).

Definition 2.13. A *weak n-category* is an opetopic set such that every niche has a universal occupant and composites of universal cells are universal.

3. Some Applications of Higher Category Theory

As has already been mentioned in the introduction, there exists applications of higher category theory to many active areas of mathematics. To geometry there is so-called 'higher geometry' (in particular *homotopical algebraic geometry* [TV04, TV06] concerns itself with certain kinds of $(\omega, 1)$-categories[4] which generalise algebraic geometry, derived algebraic geometry and spectral algebraic geometry - through studying the category of sets, the $(\omega, 1)$-category of simplicial commutative rings, and the $(\omega, 1)$-category of spectra, respectively). In the intersection of mathematical logic and category theory there is *homotopy type theory* (cf. [Sch16, Uni13]) which deals with $(\omega, 0)$-categories (also known as ∞-*groupoids*) in an effort to interpret *types*, which initially seem to be objects cataloguing certain data of variables according to rules of formation, elimination, introduction, etc. but are now considered as ∞-groupoids due to their nontrivial iterative structure (this is particularly evident in the case of the *identity type*[5], which is explained at the beginning of [Uni13, Chapt. 2]).

Of course there is the application to category theory itself: $(n+1)$-categories can be understood through the study of n-categories. The category n**Cat** has as objects the n-categories, has as 1-morphisms the n-functors (which are functors between n-categories), has as 2-morphisms the natural transformations (between n-functors), has as 3-morphisms the modifications (between natural transformations), etc. This is a strict $(n+1)$-category; one can also consider the weak $(n+1)$-category which for $n = 2$ is known as **BiCat**, the category of bicategories. The morphisms in this tricategory are *pseudofunctors*, which might not necessarily preserve all the structure between bicategories but do so 'up to equivalence' (in this case, up to an invertible 2-morphism which in turn satisfies some coherence laws). Bicategories themselves are useful objects of study, as [Lan00] expounds. Landsman demonstrates one can form a bicategory of rings (with bimodules as morphisms), a bicategory of C^*-algebras (with Hilbert bimodules as morphisms), and a bicategory of integrable

[4]An (ω, k)-category is an ω-category (an n-category for arbitrarily high n) with all m-morphisms *equivalences* (in the higher-morphism sense) for $m > k$.

[5]Given a type A, "the identity type of x and y over A", written $\mathrm{id}_A(x, y)$, can be considered as the collection of pieces of 'evidence' that x is the same as y, where x, y are variables of type A.

Poisson manifolds (with regular symplectic bimodules as morphisms - see [Lan00, §8]).

The other main application of higher category theory we will mention is to algebraic topology in the form of *homotopy theory.*

Definition 3.1. Given two topological spaces X, Y and two continuous maps f, g between them, a *homotopy* from f to g is a continuous map $h : [0,1] \times X \to Y$ such that

$$h(0, -) = f, \qquad h(1, -) = g.$$

Moreover two continuous maps f and g are called *homotopy equivalent* if there are homotopies between fg^{-1} and id_Y, and $f^{-1}g$ and id_X.

This is to say we have turned the category **Top** of topological spaces into a 2-category with homotopies acting as the 2-morphisms. Of course, we can take homotopies between homotopies, and homotopies between homotopies between homotopies, etc. which amounts to viewing **Top** now as an ω-category. The study of homotopy theory should benefit higher category theory, and vice versa, which is exactly what Grothendieck [Gro83] argued. Grothendieck pursued weak ω-functors between **Top** and the category ∞**Grp** of weak ∞-groupoids, aiming to show **Top** and ∞**Grp** are equivalent as objects of ω**Cat**. Alas, a fundamental obstruction to this conjecture is that any solution is highly dependant on what one takes as the definition of 'weak ∞-groupoid', which leads back to our earlier comments on the multitude of definitions for *n-category* and one's belief in their equivalence. For instance, if one believes Henry's definition [Hen16] of ∞-groupoid is equivalent to every other definition, then one can prove true this homotopy hypothesis of Grothendieck ([Hen16, Theorem 5.2.12]).

The final mention of applications of higher category theory we shall make are those to physics: Baez & Lauda [BL09] have an extensive catalogue of contributions of higher category theory to what they call "n-categorical physics" in the form of Feynman diagrams (ibid. p. 15), string theory (ibid. p. 40), loop quantum gravity (ibid. p. 73), and topological quantum field theory, amongst others. On this last subject Baez details (in [Bae97, §3.3]) how higher category theory can be used to understand the collection of n-dimensional manifolds embedded in $(n+k)$-dimensional spacetime, by describing these manifolds using "n-categories with duals" (a very thorough introduction on how this goes is given in [BD04]). Also in this area Baez has worked on "2-Hilbert Spaces" [Bae96] which are Hilbert-space-like categories obtained by "categorifying the various ingredients in the definition of Hilbert space" ([Bae96, §1]). These are fundamentally relevant to quantum field theory as Hilbert spaces are used to represent 'states' of a quantum system.

4. Conclusion

In this essay we have given one (of many) definitions of an *n-category* and seen some applications of higher category theory to mathematics and physics. As Baez remarks in [Bae97, §5], there are still preliminary tasks to be completed regarding the verification that the popular approaches to the definition of 'n-category' are all equivalent; once this is done, he says, "there should be many exciting things we can do with n-categories". To conclude this essay the author would like to pen some thoughts regarding a future application of higher category theory.

Category theory itself is an extremely wide-reaching area of mathematics, stretching all the way to natural language processing. In their landmark paper, Coecke et. al. [C$^+$10] propose a categorical framework for unifying the meaning of words in a corpus with the algebraic structure of grammar, to compose the full meaning of a given sentence. Essentially the authors use the product of **FVect** (the category of $\mathbb{R}$-vector spaces) and **Preg** (a partially ordered set, viewed as a category, with specific types corresponding to nouns, verbs, etc. as objects) to combine semantics and syntactics in a compositional fashion.

Specifically, if one has the sentence "Alice likes Bob" one can determine the *meaning* of this sentence by understanding the meaning of "Alice", "likes", "Bob" (which are vectors in some vector space) and the grammar (which is a reduction[6] of types $nvn \vdash s$ according to some given rules) and how these preform in the sentence together. If N is an $\mathbb{R}$-vector space of nouns and V an $\mathbb{R}$-vector space of verbs, then the meaning of the sentence "Alice likes Bob" is given by a *meaning morphism* in **FVect**$\times$**Preg**, from the total data of the words to their aggregate meaning:

$$\underset{\overrightarrow{Alice}\ \overrightarrow{likes}\ \overrightarrow{Bob}}{(N \otimes V \otimes N, nvn)} \xrightarrow{(f,\ \vdash)} \underset{\overrightarrow{Alice\ likes\ Bob}}{(S, s)},$$

where S is the 'meaning space' of sentences and s is a type returned when the composed sentence is grammatically correct. This explanation of the relevance of category theory to natural language processing has been rather terse, but fortunately [C$^+$10] gives a gentle introduction and a detailed procedure for exactly calculating the meaning of a sentence (cf. [C$^+$10, §3.5]) and the author has examples of this principle in action, in [Tyr18, §3].

This is relevant to the topic of higher categories as it may be possible to view **FVect**$\times$**Preg** as a 2-category and possibly calculate the meaning of *ambiguous sentences*. Recall that *homonyms* are words that are spelled and pronounced the same, yet have different meanings. Now consider two *homonymous sentences*, which are composed of the same words in the same order yet have different meanings (such as "Grothendieck devised many schemes", which changes depending on the meaning of the word "scheme"). Such a sentence is ambiguous as its meaning is unclear until its context is examined. Adding a 2-categorical structure to **FVect**$\times$**Preg** is therefore useful as 2-morphisms in **FVect**$\times$**Preg** can determine when meaning morphisms are equivalent. In the example

$$\Sigma \quad = \quad \text{"Grothendieck devised many schemes"} \quad = \quad \text{"}w_1\ w_2\ w_3\ w_4\text{"},$$

where w_i is an element of the vector space W_i (a word), and Σ has grammar type reduction $p_1p_2p_3p_4 \vdash s$, there are two meaning morphisms $(f_{\text{"plan"}},\ \vdash)$ and $(f_{\text{"structure"}},\ \vdash)$ for the corresponding two meanings of the word "scheme":

$$(W_1 \otimes W_2 \otimes W_3 \otimes W_4, p_1p_2p_3p_4) \underset{(f_{\text{"structure"}},\ \vdash)}{\overset{(f_{\text{"plan"}},\ \vdash)}{\rightrightarrows}} (S, s) \qquad \Downarrow E$$

The 2-morphism $E : (f_{\text{"plan"}},\ \vdash) \Rightarrow (f_{\text{"structure"}},\ \vdash)$ can be used to show $(f_{\text{"plan"}},\ \vdash)$ is not equivalent to $(f_{\text{"plan"}},\ \vdash)$, assuming E is built from data which distinguishes the two meanings of "scheme". Of course, depending on the context an ambiguous sentence may remain ambiguous, in which case E would be an equivalence. These 2-morphisms could also have applications to language translation as they serve to compare the meaning morphisms of sentences up to equivalence, which is independent of the underlying language(s). (This is the motivation for [Tyr18], and more generally [B$^+$18].)

In either occurrence, higher category theory is used to take the context of a sentence and provide information about that sentence, which is the next natural step in developing this "distributional" model of meaning.

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[6]We are slightly abusing notation, using "$\vdash$" for "reduction".

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Mathematical Institute, Woodstock Road, Oxford OX2 6GG
E-mail address: brian.tyrrell@maths.ox.ac.uk