Game Theory Handbook - CTYI Session I

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Contents

1 Introduction - Some Simple Games

Source: Wikipedia.

1.1 Two Thirds The Average

I want everyone in the class to pick a real number between zero and a hundred, inclusive. The winner is the person who gets closest to two thirds of the average of these numbers. First do this without thinking about the numbers at hand - just do it with your gut. What were our results?

- Why do you think we got the results we did?
- What strategies were you using to win?
- Were any given strategies suboptimal - could people have done better?

Lets replay the game another time.

- Have results changed with brief repetition?
- Are we learning as we go or is there another motive?

1.2 Euro Auction

I, the auctioneer, have one euro. You all can bid on it - in an *ascending auction* style (think of a typical tv auction with a usual auctioneer "going once, twice etc."). One twist - both the winner and runner up pay me for it!

- What happened?
- What should people have done?
- What should have happened?

1.3 Ultimatum Game

Split up into pairs. I will randomly select one of each of you to be the giver, and one the receiver. The giver has 1 euro - they can elect to give any amount to the receiver. If the receiver accepts - the trade happens. If they reject, it doesn't.

- What is the givers optimal choice?
- What is the receivers optimal choice?
- What really happens?

1.4 The History of Game Theory

These are all part of a wider field of mathematical economics called Game Theory. I wouldn't normally recommend Wikipedia as a source but here it actually does quite a decent job:

"Game theory is the study of strategic decision making. Specifically, it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers." An alternative term suggested "as a more descriptive name for the discipline" is 'interactive decision theory'. Game theory is mainly used in economics, political science, and psychology, as well as logic, computer science, and biology. The subject first addressed *zero-sum games*, such that one person's gains exactly equal net losses of the other participant or participants. Today, however, game theory applies to a wide range of behavioural relations, and has developed into an umbrella term for the logical side of decision science, including both humans and non-humans (e.g. computers, animals)."

In essence it is a theory of decision making. We attempt to model real life situations in a somewhat analytical way - to put it in a very crude form "make the best possible decision given everyone else's possible decisions".

1.5 A Chronology of Game Theory

(Source: Open Options Corporation.) Game theory is a well developed field of study that has attracted some of the world's greatest mathematicians, won two Nobel Prizes and is even credited with winning the Cold War.

The origins of game theory go far back in time. Recent work suggests that the division of an inheritance described in the Talmud[1] (in the early years of the first millennium) predicts the modern theory of cooperative games and, in 1713, James Waldegrave wrote out a strategy for a card game that provided the first known solution to a two player game.

Despite these early efforts, the book *The Theory of Games and Economic Behaviour* by John von Neumann and Oskar Morgenstern (published in 1944) is usually credited as the origin of the formal study of game theory. This pioneering work focused on finding unique strategies that allowed players to minimize their maximum losses ("minimax solution") by considering, for every possible strategy of their own, all the possible responses of other players. Building upon von Neumann's earlier work on two player games where the winnings of one player are equal and contrary to the losses of his opponent (zero-sum) and where each player knows the strategies available to all players and their consequences (perfect information), von Neumann and Morgenstern extended the minimax theorem to include games involving imperfect information and games with more than two players.

The golden age of game theory occurred in the 1950s and 1960s when researchers focused on **finding sets of strategies, known as equilibria, to 'solve' a game if all players behaved rationally**. The most famous of these is the *Nash equilibrium* proposed by John Nash, later made famous in the film *A Beautiful Mind* starring Russell Crowe. A *Nash equilibrium* exists if **no** player can unilaterally move to improve their own outcome. In other words, they have no incentive to change, since their strategy is the best they can do given the actions of the other players.

Nash also made significant contributions to bargaining theory and examined cooperative games where threats and promises are fully binding and enforceable. In 1965, Reinhard Selten introduced the concept of *subgame perfect equilibria*, which describes strategies that deliver Nash equilibrium across every sequential subgame of the original game[2]. Such subgame perfect equilibria may be found by first determining optimal action of the player who makes the last move of the game. Then, the optimal action of the next to last moving player is determined assuming the last player's action as given. The process, known as backward induction, continues until all players' actions have been determined. In 1967, John Harsanyi formalized Nash's work and developed incomplete information games. He, along with John Nash and Reinhard Selten, won the Nobel Prize for Economics in 1994.

[1] Which can be considered the central text of Judaism.

[2] Don't Panic! These terms will be explained later in the course.

Another important contribution to game theory during the 1950s and 1960s was Luce and Raiffa's book, *Games and Decisions*. *The Prisoner's Dilemma*, introduced by the RAND Corporation and very familiar to any MBA student, is also a product of this period.

Further adding to the acclaim of game theory, another Nobel Prize was awarded to game theorists, Robert Aumann and Thomas Schelling, in 2005. Schelling used game theory in his 1960 book *The Strategy of Conflict* to explain why credible threats of nuclear annihilation from the U.S. and the (former) Soviet Union were counterbalancing through mutually assured destruction and therefore were not likely to be used. He also argued that the ability to retaliate was more useful that the ability to withstand an attack.

Aumann's work was mathematical and focused on whether co-operation increases if games are continually repeated rather than played out in a single encounter. He showed that co-operation is less likely when there are many participants, when interactions are infrequent, when the time horizon is short or when others' actions cannot be clearly observed (i.e. we don't get a chance to learn from our mistakes and overcome our mistrust of strangers).

Throughout the years, game theory has been applied to many different fields of study including auction of underused radio spectra, artificial intelligence, bargaining, evolutionary biology, political science and real world business decisions.

1.6 John von Neumann (28/12/1903 - 08/02/1957)

John von Neumann was born 'Neumann János Lajos' in Hungary, 1903. He was called Jancsi as a child, a diminutive form of János, then later became 'Johnny' in the United States. His father, Max Neumann, was a top banker and he was brought up in a extended family, living in Budapest where as a child he learnt languages from the German and French governesses that were employed. Although the family were Jewish, Max Neumann did not observe the strict practices of that religion and the household seemed to mix Jewish and Christian traditions.

It is also worth explaining how Max Neumann's son acquired the "von" to become János von Neumann. In 1913 Max Neumann purchased a title, but did not change his name. His son, however, used the German form 'von Neumann' where the "von" indicated the title.

As a child von Neumann showed he had an incredible memory. William

Poundstone wrote:

"At the age of six, he was able to exchange jokes with his father in classical Greek. The Neumann family sometimes entertained guests with demonstrations of Johnny's ability to memorise phone books. A guest would select a page and column of the phone book at random. Young Johnny read the column over a few times, then handed the book back to the guest. He could answer any question put to him (who has number such and such?) or recite names, addresses, and numbers in order."

In 1911 von Neumann entered the Lutheran Gymnasium, known as *Fasori Evangelikus Gimnázium*. The school had a strong academic tradition which seemed to count for more than the religious affiliation both in the Neumann's eyes and in those of the school. His mathematics teacher quickly recognised von Neumann's genius and special tuition was put on for him. The school had another outstanding mathematician one year ahead of von Neumann, namely Eugene Wigner (a theoretical physicist who recieved the Nobel Prize in Physics in 1963). A few years after von Neumann, Paul Erdös (b. 1913), another world class mathematician, attended the school.

World War I had relatively little effect on von Neumann's education but, after the war ended, Béla Kun controlled Hungary for five months in 1919 with a Communist government. The Neumann family fled to Austria as the affluent came under attack. However, after a month, they returned to face the problems of Budapest. When Kun's government failed, the fact that it had been largely composed of Jews meant that Jewish people were blamed. Such situations are devoid of logic and the fact that the Neumann's were opposed to Kun's government did not save them from persecution.

In 1921 von Neumann completed his education at the Lutheran Gymnasium. His first mathematics paper, written jointly with Fekete; the assistant at the University of Budapest who had been tutoring him, was published in 1922. However Max Neumann did not want his son to take up a subject that would not bring him wealth (Hungary had very few postings for professional mathematicians at the time). Max Neumann asked Theodore von Kármán to speak to his son and persuade him to follow a career in business. Perhaps von Kármán was the wrong person to ask to undertake such a task but in the end all agreed on the compromise subject of chemistry for von Neumann's university studies.

Hungary was not an easy country for those of Jewish descent for many reasons and there was a strict limit on the number of Jewish students who could enter Pázmány Péter University, AKA the University of Budapest. Of course, even with a strict quota, von Neumann's record easily won him a place to study mathematics in 1921 but he did not attend lectures. Instead he also entered the University of Berlin in 1921 to study chemistry.

Von Neumann studied chemistry at the University of Berlin until 1923 when he went to Zürich. He achieved outstanding results in the mathematics examinations at the University of Budapest despite not attending any courses. Von Neumann received his diploma in chemical engineering from the Technische Hochschule in Zürich in 1926 (now known as ETH Zürich). While in Zürich he continued his interest in mathematics, despite studying chemistry, and interacted with Hermann Weyl and George Pólya who were both at Zürich. He even took over one of Weyl's courses when he was absent from Zürich for a time. Pólya said:

"Johnny was the only student I was ever afraid of. If in the course of a lecture I stated an unsolved problem, the chances were he'd come to me as soon as the lecture was over, with the complete solution in a few scribbles on a slip of paper."

Von Neumann received his doctorate in mathematics from the University of Budapest, also in 1926, with a thesis on the axiomatization of Cantor's set theory (set theory is a very fundamental branch of mathematics). He published a definition of ordinal numbers[3] when he was 19; the definition used today.

1.7 Oskar Morgenstern (24/1/1902 - 26/07/1977)

Morgenstern was born on January 24, 1902 in Görlitz, Germany. His mother was an illegitimate daughter of German emperor, Frederick III. He graduated from the University of Vienna, earning a doctorate in political science in 1925. He received a scholarship from the Rockefeller Foundation to further his studies in the United States, where he spent the next four years.

Upon his return to Austria in 1929, Morgenstern started work at the University of Vienna, first as a lecturer and then a professor in economics. During that time he belonged to the so-called "Austrian circus," a group of Austrian economists including Gottfried Haberler and Friedrich von Hayek, who met regularly with Ludwig von Mises to discuss different issues in the field. The group was the Austrian equivalent of Keynes's "Cambridge Circus."

In 1938 Morgenstern traveled to the United States as a visiting professor in

[3] A concept related to set theory

economics at Princeton University in New Jersey. It was there that he heard the news that Adolf Hitler had occupied Vienna, and that it would probably be unwise to return to Austria. Morgenstern decided to stay in the United States, becoming a naturalized citizen in 1944.

After Morgenstern became a member of the faculty at Princeton he started to work closely with mathematician John von Neumann, developing a theory of predicting economic behavior. In 1944, they wrote *Theory of Games and Economic Behavior*, recognized as **the first book on game theory**.

Morgenstern married Dorothy Young in 1948.

Throughout the 1950s and 1960s Morgenstern continued to write on different economic issues, publishing *On the Accuracy of Economic Observations* in 1950, *Prolegomena to a Theory of Organization* in 1951, and *The Question of National Defense and International Transactions and Business Cycles* in 1959. He retired from Princeton in 1970.

Morgenstern accepted the position of professor in economics at New York University in 1970, where he remained until his death in 1977. New York University appointed Morgenstern its distinguished professor of game theory and mathematical economics, a position that really hadn't existed before his time, just before his death.

Morgenstern died in Princeton, New Jersey, on July 26, 1977.

1.8 John Forbes Nash JR (13/06/1928 - 23/05/2015)

Famously had a film made about his life - we will try get round to watching some of this at some point! In the meantime, I strongly suggest having a read of his biography, *A Beautiful Mind* by Sylvia Nasar.

John F. Nash's father, also called John Forbes Nash (we shall refer to him as John Nash Senior), was a native of Texas. John Nash Senior was born in 1892 and had an unhappy childhood from which he escaped when he studied electrical engineering at Texas Agricultural and Mechanical. After military service in France during World War I, John Nash Senior lectured on electrical engineering for a year at the University of Texas before joining the Appalachian Power Company in Bluefield, West Virginia. John F Nash's mother, Margaret Virginia Martin, was known as Virginia. She had a university education, studying languages at the Martha Washington College and then at West Virginia University. She was a school teacher for ten years before meeting John Nash Senior, and the two were married on 6 September

1924.

Johnny Nash, as he was called by his family, was born in Bluefield Sanitarium and baptised into the Episcopal Church. He was

"a singular little boy, solitary and introverted"

but he was brought up in a loving family surrounded by close relations who showed him much affection. After a couple of years Johnny became an older brother when Martha was born. He seems to have shown a lot of interest in books when he was young but little interest in playing with other children. It was not because of lack of children that Johnny behaved in this way, for Martha and her cousins played the usual childhood games: cutting patterns out of books, playing hide-and-seek in the attic, playing football. However while the others played together Johnny played by himself with toy airplanes and matchbox cars.

His mother responded by enthusiastically encouraging Johnny's education, both by seeing that he got good schooling and also by teaching him herself. Johnny's father responded by treating him like an adult, giving him science books when other parents might give their children colouring books.

Johnny's teachers at school certainly did not recognise his genius, and it would appear that he gave them little reason to realise that he had extraordinary talents. They were more conscious of his lack of social skills and, because of this, labelled him as 'backward'. It now would appear that he was extremely bored at school. By the time he was about twelve years old he was showing great interest in carrying out scientific experiments in his room at home. It is fairly clear that he learnt more at home than he did at school.

Martha seems to have been a remarkably normal child while Johnny seemed different from other children. She wrote later in life

"Johnny was always different. [My parents] knew he was different. And they knew he was bright. He always wanted to do things his way. Mother insisted I do things for him, that I include him in my friendships. ... but I wasn't too keen on showing off my somewhat odd brother."

His parents encouraged him to take part in social activities and he did not refuse, but sports, dances, visits to relatives and similar events he treated as tedious distractions from his books and experiments.

Nash first showed an interest in mathematics when he was about 14 years old. Quite how he came to read E. T. Bell's *Men of Mathematics* is unclear but certainly this book inspired him. He tried, and succeeded, in proving for himself results due to Fermat which Bell stated in his book. The excitement that Nash found here was in contrast to the mathematics that he studied at school which failed to interest him.

He entered Bluefield College in 1941 at age 13 and there he took mathematics courses as well as science courses, in particular studying chemistry, which was a favourite topic. He began to show abilities in mathematics, particularly in problem solving, but still with hardly any friends and behaving in a somewhat eccentric manner, this only added to his fellow pupils view of him as peculiar. He did not consider a career in mathematics at this time, however, which is not surprising since it was (and to some degree still is...) an unusual profession. Rather he assumed that he would study electrical engineering and follow his father but he continued to conduct his own chemistry experiments and was involved in making explosives which led to the death of one of his fellow pupils:

"Boredom and simmering adolescent aggression led him to play pranks, occasionally ones with a nasty edge.

He caricatured classmates he disliked with weird cartoons, enjoyed torturing animals, and once tried to get his sister to sit in a chair he had wired up with batteries."

Nash won a scholarship in the George Westinghouse Competition and was accepted by the Carnegie Institute of Technology (now Carnegie-Mellon University) which he entered in June 1945 with the intention of taking a degree in chemical engineering. Soon, however, his growing interest in mathematics had him take courses on tensor calculus and relativity. There he came in contact with John Synge, the Irish mathematician and physicist, who had recently been appointed as Head of the Mathematics Department and taught the relativity course. Synge and the other mathematics professors quickly recognised Nash's remarkable mathematical talents and persuaded him to become a mathematics specialist. They realised that he had the talent to become a professional mathematician and strongly encouraged him.

Nash quickly aspired to great things in mathematics. He took the William Lowell Putnam Mathematics Competition (maths competition across the US & Canada for a giant scholarship) twice but, although he did well, he did not make the top five. It was a failure in Nash's eyes and one which he took badly. The Putnam Mathematics Competition was not the only thing going badly for Nash. Although his mathematics professors heaped praise on him, his fellow students found him a very strange person. Physically he was strong and this saved him from being bullied, but his fellow students took delight in making fun of Nash who they saw as an awkward immature person displaying childish tantrums. One of his fellow students wrote:-

“He was a country boy unsophisticated even by our standards. He behaved oddly, playing a single chord on a piano over and over, leaving a melting ice cream cone melting on top of his cast-off clothing, walking on his roommate’s sleeping body to turn off the light.”

Another wrote:-

“He was extremely lonely.”

And a third fellow student wrote:-

“We tormented poor John. We were very unkind. We were obnoxious. We sensed he had a mental problem.”

Nash received a BA and an MA in mathematics in 1948. By this time he had been accepted into the mathematics programme at Harvard, Princeton, Chicago and Michigan. He felt that Harvard was the leading university and so he wanted to go there, but on the other hand their offer to him was less generous than that of Princeton. Nash felt that Princeton were keen that he went there while he felt that his lack of success in the Putnam Mathematics Competition meant that Harvard were less enthusiastic. He took a while to make his decision, while he was encouraged by Synge and his other professors to accept Princeton. When Lefschetz offered him the most prestigious Fellowship that Princeton had, Nash made his decision to study there.

In September 1948 Nash entered Princeton where he showed an interest in a broad range of pure mathematics: topology, algebraic geometry, game theory and logic were among his interests but he seems to have avoided attending lectures. Usually those who decide not to learn through lectures turn to books but this appears not to be so for Nash, who decided not to learn mathematics ”second-hand” but rather to develop topics himself. In many ways this approach was successful for it did contribute to him developing into one of the most original of mathematicians who would attack a problem in a totally novel way.

In 1949, while studying for his doctorate, he wrote a paper which 45 years later was to win a Nobel prize for economics. During this period Nash established the mathematical principles of game theory. P. Ordeshook wrote:

“The concept of a *Nash equilibrium n-tuple* is perhaps the most important idea in noncooperative game theory. ... Whether we are analysing candidates’ election strategies, the causes of war, agenda manipulation in legislatures, or the actions of interest groups, predictions about events reduce to a search for and description of equilibria. Put simply, equilibrium strategies are the things that we predict about people."

Milnor, who was a fellow student, describes Nash during his years at Princeton:

"He was always full of mathematical ideas, not only on game theory, but in geometry and topology as well. However, my most vivid memory of this time is of the many games which were played in the common room. I was introduced to Go and Kriegspiel, and also to an ingenious topological game which we called *Nash* in honour of the inventor."

In fact the game "Nash" was almost identical to *Hex* which had been invented independently by Piet Hein in Denmark. In a few days time, we will play a few games of *Hex* and discuss some mathematical results around the game.

Here are four comments from fellow students:

"Nash was out of the ordinary. If he was in a room with twenty people, and they were talking, if you asked an observer who struck you as odd it would have been Nash. It was not anything he consciously did. It was his bearing. His aloofness.

Nash was totally spooky. He wouldn't look at you. he'd take a lot of time answering a question. If he thought the question was foolish he wouldn't answer at all. He had no affect. It was a mixture of pride and something else. He was so isolated but there really was underneath it all a warmth and appreciation of people.

A lot of us would discount what Nash said. ... I wouldn't want to listen. You didn't feel comfortable with the person.

He had ideas and was very sure they were important. He went to see Einstein not long after he arrived in Princeton and told him about an idea he had regarding gravity. After explaining complicated mathematics to Einstein for about an hour, Einstein advised him to go and learn more physics. Apparently a physicist did publish a similar idea some years later."

In 1950 Nash received his doctorate from Princeton with a thesis entitled *Non-cooperative Games.* In the summer of that year he worked for the RAND Corporation where his work on game theory made him a leading expert on the Cold War conflict which dominated RAND's work. He worked there from time to time over the next few years as the Corporation tried to apply game theory to military and diplomatic strategy. Back at Princeton in the autumn of 1950 he began to work seriously on pure mathematical problems. It might seem that someone who had just introduced ideas which would, one day, be considered worthy of a Nobel Prize would have no problems finding an academic post. However, Nash's work was not seen at the time to be of outstanding importance and he saw that he needed to make his mark in other ways. We should also note that it was not really a **move** towards pure mathematics for he had always considered himself a pure mathematician. He had already obtained results on manifolds and algebraic varieties before writing his thesis on game theory. His famous theorem, that any compact real manifold is diffeomorphic to a component of a real-algebraic variety, was thought of by Nash as a possible result to fall back on if his work on game theory was not considered suitable for a doctoral thesis. He said in a recent interview:-

"I developed a very good idea in pure mathematics. I got what became [the paper] Real Algebraic Manifolds. I could have published that earlier, but it wasn't rushed to publication. I took some time in writing it up. Somebody suggested that I was a prodigy. Another time it was suggested that I should be called "bug brains", because I had ideas, but they were sort of buggy or not perfectly sound. So that might have been an anticipation of mental problems. I mean, taking it at face value."

In 1952 Nash published *Real Algebraic Manifolds* in the Annals of Mathematics. The most important result in this paper is that two real algebraic manifolds are equivalent if and only if they are analytically homeomorphic. Although publication of this paper on manifolds established him as a leading mathematician, not everyone at Princeton was prepared to see him join the Faculty there. This was nothing to do with his mathematical ability which everyone accepted as outstanding, but rather some mathematicians such as Artin felt that they could not have Nash as a colleague due to his aggressive personality.

Halmos received the following letter in early 1953 from Warren Ambrose relating to Nash:-

"There's no significant news from here, as always. Martin is appointing John Nash to an Assistant Professorship (not the Nash at Illinois, the one out of Princeton by Steenrod) and I'm pretty annoyed at that. Nash is a childish bright guy who wants to be "basically original," which I suppose is fine for those who have some basic originality in them. He also makes a damned fool of himself in various ways contrary to this philosophy. He recently heard of the unsolved problem about imbedding a Riemannian manifold isometrically in Euclidean space, felt that this was his sort of thing, provided the problem were sufficiently worthwhile to justify his efforts; so he proceeded to write to everyone in the math society to check on that, was told that it probably was, and proceeded to announce that he had solved it, modulo details, and told Mackey he would like to talk about it at the Harvard colloquium. Meanwhile he went to Levinson to inquire about a differential equation that intervened and Levinson says it is a system of partial differential equations and if he could only [get] to the essentially simpler analog of a single ordinary differential equation it would be a damned good paper - and Nash had only the vaguest notions about the whole thing. So it is generally conceded he is getting nowhere and making an even bigger ass of himself than he has been previously supposed by those with less insight than myself. But we've got him and saved ourselves the possibility of having gotten a real mathematician. He's a bright guy but conceited as Hell, childish as Wiener, hasty as X, obstreperous as Y, for arbitrary X and Y."

Ambrose, the author of this letter, and Nash had rubbed each other the wrong way for a while. They had played silly pranks on each other and Ambrose seems not to have been able to ignore Nash's digs in the way others had learned to do. It had been Ambrose who had said to Nash:-

"If you're so good, why don't you solve the embedding theorem for manifolds?"

Collective 'OOOOOOooooooooooo' please. Sidenote: he did, in part, the year later in 1954.

From 1952 Nash had taught at the Massachusetts Institute of Technology but his teaching was unusual (and unpopular with students) and his examining methods were 'highly unorthodox'.

He met Eleanor Stier and they had a son, John David Stier, who was born on 19 June 1953. Eleanor was a shy girl, lacking confidence, a little afraid of men, did not want to be involved. She found in Nash someone who was even less experienced than she was and found that attractive.

Nash was looking for emotional partners who were more interested in giving than receiving, and Eleanor was apparently very much that sort.

Nash did not want to marry Eleanor although she tried hard to persuade him. In the summer of 1954, while working for RAND, Nash was arrested for indecent exposure in a police operation to trap homosexuals. He was dismissed from RAND.

On 4 January he was back at the university and started to teach his game theory course. His opening comments to the class were:-

"The question occurs to me. Why are you here?"

One student immediately dropped the course. Nash asked a graduate student to take over his course and vanished for a couple of weeks. When he returned he walked into the common room with a copy of the New York Times saying that it contained encrypted messages from outer space that were meant only for him. For a few days people thought he was playing an elaborate private joke.

Norbert Wiener was one of the first to recognise that Nash's extreme eccentricities and personality problems were actually symptoms of a medical disorder. After months of bizarre behaviour, Alicia had her husband involuntarily hospitalised at McLean Hospital, a private psychiatric hospital outside of Boston. Upon his release, Nash abruptly resigned from MIT, withdrew his pension, and went to Europe, where he intended to renounce his US citizenship. Alicia left her newborn son (named John Charles Martin Nash) with her mother, and followed the ill Nash. She then had Nash deported - back to the United States.

After their return, the two settled in Princeton where Alicia took a job. Nash's illness continued, transforming him into a frightening figure. He spent most of his time hanging around on the Princeton campus, talking about himself in the third person as Johann von Nassau, writing nonsensical postcards and making phone calls to former colleagues. They stoically listened to his endless discussions of numerology and world political affairs. Her husband's worsening condition depressed Alicia more and more.

In January 1961 the despondent Alicia, John's mother, and his sister Martha made the difficult decision to commit him to Trenton State Hospital in New Jersey where he endured insulin-coma therapy, an aggressive and risky treatment, five days a week for a month and a half. A long sad episode followed which included periods of hospital treatment, temporary recovery, then further treatment. Alicia divorced Nash in 1962. Nash spent a while with Eleanor and John David. In 1970 Alicia tried to help him taking him in as a boarder, but he appeared to be lost to the world, removed from ordinary society, although he spent much of his time in the Mathematics Department at Princeton.

In the 1990s Nash made a recovery from the schizophrenia from which he had suffered since 1959. He and Alicia resumed their relationship and remarried in 2001. His ability to produce mathematics of the highest quality did not totally leave him. He said:-

"I would not treat myself as recovered if I could not produce good things in my work."

Nash was awarded (jointly with Harsanyi and Selten) the 1994 Nobel Prize in Economic Science for his work on game theory. In 1999 he was awarded the Leroy P Steele Prize by the American Mathematical Society:-

"for a seminal contribution to research"

Nash and Louis Nirenberg were awarded the Abel prize in 2015 for:

"... striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis."

A few days after picking up the prize in Norway, Nash and his wife Alicia were killed in an accident in a taxi on the New Jersey turnpike.

1.9 Intuitive Examples

So what exactly can we infer about the contributions of these people to not just maths - but the study of Game Theory? The first we can see in a famous video from "A Beautiful Mind".

To summarise it:

There are five of them in a bar and five girls walk in - they all have interest in one specific girl, but why don't they go for her? It all has roots in Game Theory and the concept of a "Nash Equilibrium".

Definition. A Nash Equilibrium in a simultaneous game is a situation in which no player in a game finds it optimal to switch strategies, given the strategies of the other players - "a long term steady state".

In layman's terms, in a Nash equilibrium no one wishes that they made a different choice when they observe the outcome.

Aside... Why isn't the situation modelled in the film a Nash equilibrium?! (There are several reasons! **Hint:** note that John Nash's work is in noncooperative game theory.)

Another example is the famous *Prisoners dilemma* - Wikipedia has a nice explaination. We will explore this in more detail at a later date from a mathematical perspective but it's easy to illustrate here.

Suppose two rivals, Alice and Bob, have been arrested for gang activity. The police know Alice and Bob belong to rival gangs, but don't have much more information than that. They have enough charges to jail both prisoners for one year, however if one prisoner snitches on the other prisoner, the other prisoner will be jailed for ten years. If both snitch, their evidence isn't as valuable so they can both be prosecuted for only five years. The police decide to strike the following deal:
A lawyer enters Alice's cell, while at the same time a lawyer enters Bob's cell. Each lawyer tells the prisoner the following: "We have evidence to lock you up for one year. However, if you snitch on Alice/Bob we will free you and imprison Alice/Bob for ten years. If you both snitch, you both get five years. Note that Alice/Bob has the same deal."

The prisoners have to make their choice, there and then. What should they do?

A true prisoner's dilemma is typically "played" only once; otherwise it is classified as an iterated prisoner's dilemma. Have a look at this video for some maths to go along with this.

1.9.1 Tic Tac Toe Example

How do you win at tic tac toe? (note: if you both play correctly you should always draw... but if you go first you have an advantage as you can capitalise if your opponent slips up - its a solved game!)

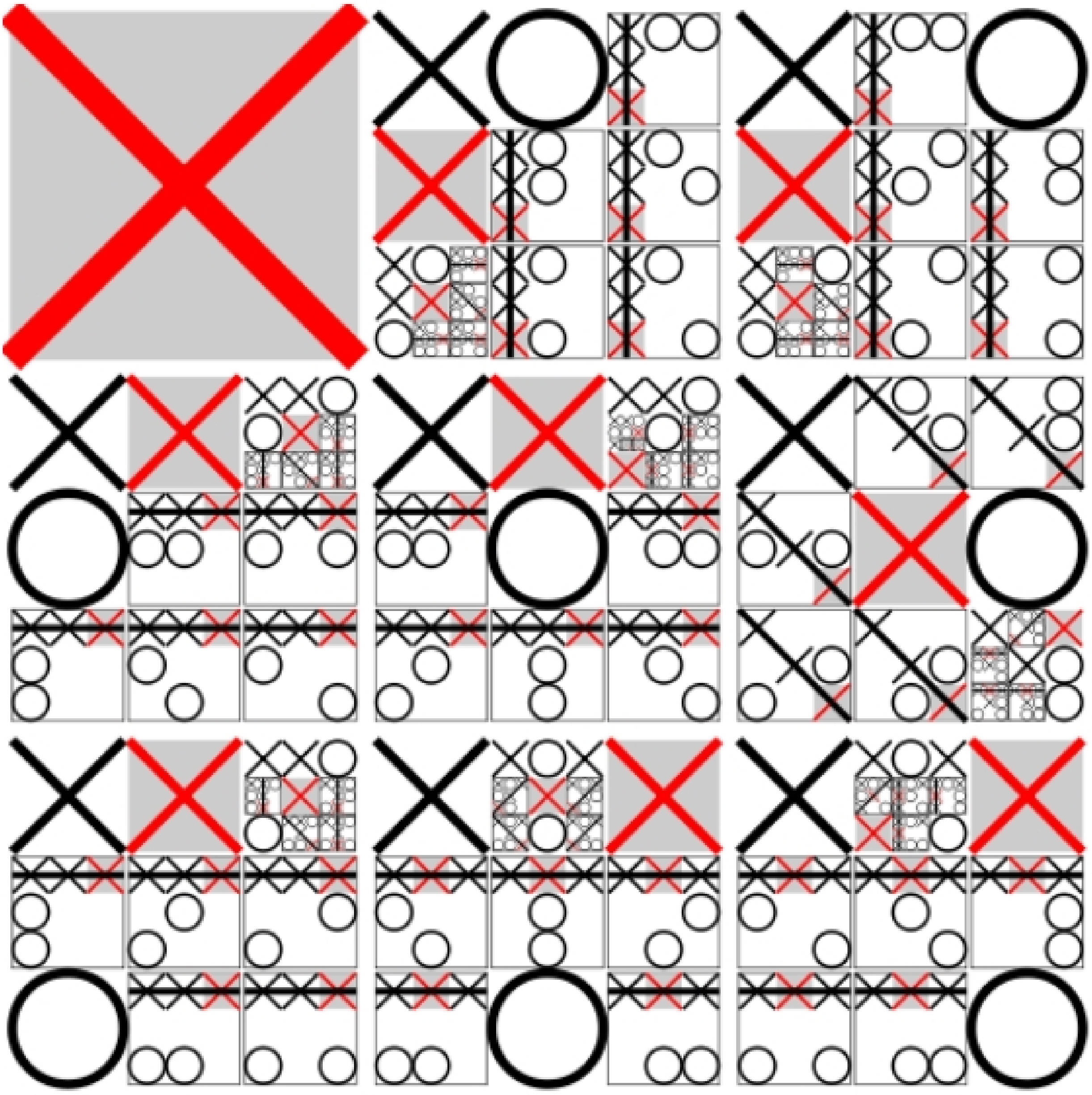


Figure 1: How to win at tic-tac-toe

Kind of a weird figure made by Numberphile but it's pretty handy. How is such a graph made? The red X shows your best move in each go and the other panels show what you should do after. So basically red X in the top left hand corner and proceed from there - I will show a better example on the board to illustrate this - but all in all this is handy.

However... What if the player playing O also knows this? What happens in the long run?

This is an example of *first mover advantage* - where else can we see this?

Another game which is commonly cited is rock paper scissors - we will come across this later.

1.9.2 The Dark Knight - Ferry Scene

Source: Michael A. Allen - "The Dark Knight and Game Theory" A weird example. How many people have seen this film? You will probably be aware of the scene I'm about to show you!

The Story

The Joker's final act as criminal mastermind and agent of nihilism involves two ferries filled with people. The first ferry is filled with normal, law abiding citizens - men, women and children - while the second ferry is filled with the population of Gotham Prison. The Joker has wired the ships with powerful explosives such that their explosion would destroy the entire ship and everyone aboard. No single individual is allowed to escape. Each ship is given a detonator for the other ferry. The use of the detonator saves the ship while killing everyone aboard the opposing ship. Thus, if any member of Ship A pushes the detonator, then Ship B is destroyed and all of Ship A is saved. Additionally, if either ship fails to use the detonator to destroy its opponent, then both ships will be destroyed by the Joker. Assuming that the actors must make their decision simultaneously, this would lead to the following game:

		Ship B	
		Cooperate	Detonate
Ship A	Cooperate	0,0	-1,1
	Detonate	1,-1	-1,-1

Solving this game is pretty straightforward and (Detonate, Detonate) becomes the dominant strategy as, at best, cooperation is 'weakly dominated'. Thus, both will ships will be destroyed.

However, in Gotham City, this does not happen - nor would it necessarily happen in a laboratory experiment either as a few more complications are introduced into the game. First, the Joker gives both ferries 30 minutes in which they can detonate the other side. Even with this complication, the outcome should be the same and both actors ought to choose detonation when the first move is available. Using *backwards deduction*, both players recognise that their opponent will choose to detonate in the final iteration even if there are some gains to short-term cooperation. This effect cascades backwards to the initial decision and mutual detonation occurs to prevent receiving the sucker's payout of cooperating while the opponent defects.

Decision Rules

Beyond this, it appears that the decision making process for both ships is different. In the ship containing prisoners, the decision to detonate becomes decentralized and any one actor willing to grab the detonator could do so. While the armed guards gives the opposite impression of clear authoritarianism, Decentralization becomes apparent as the time moves on. Thus, decentralized decision making should lead to the optimal play as any single individual among the 500 or more sub-actors should have a preference for survival.

On the civilian ship, the decision mechanism becomes a simple majority vote. When the votes are aggregated, the decision to detonate the other ferry is chosen at a rate of almost 3 to 1. Yet, there is no executive to carry out the decision and the majority will does not prevail as no single sub-actor is willing to push the button. This psychological separation between deference of active responsibility (voting to kill others) versus carrying out the task, has been illuminated in some experimental settings which I encourage you to google and explore yourselves.

The civilians act rationally as long as they, individually, are not too involved in carrying out a potentially morally reprehensible act.

Morality?

Perhaps social norms mattered for the actors in the game? In the second game, we can assume that there are some social benefits from being a *moral agent*; however, being moral is not as beneficial as being alive. Since survival trumps morality, we get another game:

The equilibria are (Cooperate, Detonate) and (Detonate, Cooperate). What has changed between this game and the last?

What if morality trumps survival? We get the following table:

		Ship B	
		Cooperate	Detonate
Ship A	Cooperate	1,1	1,2
	Detonate	2,1	0,0

		Ship B	
		Cooperate	Detonate
Ship A	Cooperate	2,2	2,1
	Detonate	1,2	0,0

Here, the Nash equilibrium is to always cooperate. However humans tend not to be very moral people, and in reality it is not true that in most people being moral is more important than survival.

What do you think is a realistic game to model this situation? Here is a blank table to input your choices:

		Ship B	
		Cooperate	Detonate
Ship A	Cooperate	,	,
	Detonate	,	,

So, what does this teach us? In reality things aren't so clear cut. What I want to challenge you to do in this course is not just understand the technical aspects to game theory, but to also understand more subtle nuances.

(Source)

2 Technical concepts

Note: It is traditional in this field to typically use female pronouns when needed.

2.1 The Decision Problem

In economics, how do we think about decisions? There are three components to a decision problem:

Definition. Problems have three features;

1. Actions - these are all the things a player can choose
2. Outcomes - the possible consequences that can result from any of these actions
3. Preferences - can describe how the player ranks the set of possible outcomes - from most to least desired. We use the symbol $\succeq$ to say that x is as good as y, i.e. $x \succeq y$.

Example: Doing an assignment for a course - the action is to do it or not, outcome is pass or fail, preferences are what you wish to do as a part of it (usually pass is preferred to fail). This is a discrete example. Continuous examples also exist - perhaps if the outcome is a continuous percentage grade in this example.

We assume two things with regards to preference relations:

1. **The completeness axiom:** (**totality**) The preference relation is complete - any two outcomes $x, y \in X$ can be ranked by the preference relation, so that either $x \succeq y$ or $y \succeq x$.
2. **The transitivity axiom:** If $x, y, z \in X$ $x \succeq y$ $y \succeq z$ then $x \succeq z$.

Combined, these define a **'rational' preference relation**.

Definition. A **payoff function** $u : X \to \mathbb{R}$ represents the preference relation $\succeq$ if for any pair $x, y \in X$ $u(x) \geq u(y) \iff x \succeq y$.

Payoff is purely **ordinal** - the numbers don't matter. If u represents a decision makers preference then any increasing function function of u also represents it.

An important component in many game theory concepts is the *theory of rational choice.* This is a component of many game theory models. A decision maker chooses the best action according to her preferences among all the actions available to her. Her rationality lies in the inconsistency of the decisions faced by her among the choices - she basically must be consistent.

So the assumptions we have are the following:

Definition. *The rational choice assumptions.*

The player fully understands the decision problem by knowing:

1. All possible actions -A.
2. All possible outcomes - X.
3. Exactly how each action affects which outcome will materialise.
4. Her rational preferences over outcomes.

A rational player facing a decision problem will choose action a if and only if it maximises her own payoff - a is chosen if and only if it's payoff is higher than any other action. In other terms:

$$a* \in A \text{ is chosen} \iff v(a*) \geq v(a) \qquad \forall a \in A.$$

2.2 Interacting Decision Makers

So far we have only looked at decision makers who choose an action from a set A, and cares only about this action. In reality a decision maker cannot control all the variables that affect her - the player may be able to determine their own action but not other actions. In other words, how do we choose an action when we don't actually know the outcomes? This is where game theory comes into play.

What is game theory?

It's a study of strategic interaction - and strategic decisions must take into account the actions of others. The key concept is *interdependence* - what I do effects you, what you do effects me. A lot of economics phenomena are linked to this. For example, pharmaceutical firms when deciding how much to invest. Members of OPEC[4] when making choices as to how to vote. Us in a group project - these are all strategic interactions. Nearly everything we do is a strategic interaction.

Definition. :

A **game** is a formal description of a strategic setting. A **solution concept** is a prescription or prediction about the outcomes of games.

We need to use different solution concepts depending on the kind of game we are looking at - we need to look beyond the regular thinking of 'what move should I make for the next turn **only**?'.

There are lots of different types of games we can think of. Particularly we can have games be different across two key dimensions - time and information.

How many moves do the players have?

- Simultaneously games (what we will be considering for the most part).
- Sequential games (which we will define and work with in examples on occasion).

Another dimension is with regards to what information the players have:

- Perfect information.
- Imperfect information.

We look at all these dimensions and try and see what is the appropriate solution concept. Let's start by playing a few games!

2.3 A Sample Game - The Grade Game

Tear off a small piece of paper, enough to write a single letter on.

[4] Organisation of the Petroleum Exporting Countries.

The Rules: Without showing the other people in the class what you're doing, write down either the letter alpha or the letter beta (when you've fully read the instructions). Each of you will be randomly paired with another person in the class. Here is how grades may be assigned:

- If you put alpha, and your partner puts beta, you get an A, your partner gets a C.
- If you both put alpha you both get a B-.
- If you both put beta, you both get a B+.
- If you put beta and your partner puts alpha, you get a C, your partner gets an A.

Your objective is to get the highest grade. Off you go!

Write down your grade on the back of your piece of paper. Let's analyse some statistics for this class.

Last year, 27% of the class chose beta, 73% chose alpha. Economics classes normally deviate more from the norm than us. What was missing from this game was preferences - the outcomes of the game really differed with regards to what type of person each participant was. Interestingly this showed we were somewhat nicer than Yale in that students there had only 15% choosing beta. In mass experiments, the split is close to 70-30 (α-β).

Issues with this game - we aren't told anything about prevailing preferences in this world. We just played it. Regardless, this technically isn't a game - we have players and strategies - i.e. possible actions, but no **well defined payoffs**.

We can have a few different kinds of combinations of player. Say we have either altruistic or selfish people in our defined universe. Altruistic people put some value on the other persons grade - probably making them more likely to pick β than alpha as they have some degree of sympathy for the opponent.

Say we are told the rates of these preferences beforehand, and the kind of people you are matched with - how does this change the game? Say a selfish person was with another predefined selfish person - they'd pick alpha! The possible scenarios as they stand are:

- Two Selfish types
- Two Altruistic Types
- Selfish vs. Altruistic
- Altruistic vs. Selfish

To analyse this we can define a matrix of possible payoffs - let's give our preferences a payoff/utility function.

Our preference order would be:

$$A \succeq B+ \succeq B- \succeq C.$$

Suppose we only care about our own grades - so let's say:

$$u(A) = 3, \quad u(B+) = 1, \quad u(B-) = 0, \quad u(C) = -1$$

Putting this into a payoff matrix:

Partner

Me		Alpha	Beta
	Alpha	0,0	3,-1
	Beta	-1,3	1,1

What would we pick if we were being purely logical? Alpha of course - no matter what the other person picks we are better off picking alpha. After all, if your partner picks alpha we get 0 rather than -1, and if she picks beta we get 3 instead of 1.

However, how may this change if cooperation is allowed, and if our functions are edited to give a sense of altruism amongst yourself and your classmates.

Let's have a read of the Grade Game Handout from Yale.

2.4 Strategic Games

Definition. A **strategic game** consists of

1. A set of players
2. For each player, a set of possible actions
3. For each player, ordinal/ordered preferences over the set of action profiles (the list of all the players actions).

Example.

- Players: Firms.
- Actions: Prices - for example, what price to set[5].
- Preferences: A reflection of the firms profits.

Example.

- Players: Candidates for political offices.
- Actions: Campaign expenditures.
- Preferences: The candidates probabilities of winning based on strategies.

As in the theory of rational choice by a single decision maker, we define a payoff function to express the players preferences ordinally.

Features of strategic games:

- Time is absent from the model - players choose their actions **simultaneously** - and the action is chosen **once and for all**.
- No player is informed when she chooses their decision of the other persons actions.

The most well known strategic game is the **Prisoners Dilemma**[6].

[5]The concept of an 'oligopoly' comes in here. An oligopoly is a state of limited competition in which a market is shared by a small number of producers or sellers. Recall a 'monopoly' too.

[6]The Wikipedia article contains more detail and variants than we will cover, so I recommend having a browse if bored.

2.4.1 The Prisoners Dilemma

Example. We have two suspects of a crime. The prosecutors lack sufficient evidence to convict the pair on the principal charge. They hope to get both sentenced to a year in prison on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain: If they both stay quiet each will receive 1 year in prison. If one snitches on the other the snitch will be freed, and the other gets 3 years. If they both snitch on the other, they both get 2 years.

This is a strategic game:

- Players: The Two Suspects.
- Actions - {Snitch, Quiet}.
- Payoff function - For Suspect 1, say, the domain of the payoff function is the following: (Snitch,Quiet) - gets freed, (Quiet,Quiet) - gets 1 year in prison, (Snitch, Snitch) - gets 2 years in prison, (Quiet,Snitch) - gets 3 years in prison. For Suspect 2, the domain is the mirror.

For Suspect 1 we set:

$$u_1(Snitch, Quiet) = 3 > u_1(Quiet, Quiet) = 2 > u_1(Snitch, Snitch) = 1 > u_1(Quiet, Snitch) = 0.$$

By construction, Suspect 2 has identical preferences to Suspect 1.

We represent the outcomes like so:

		Suspect 2	
		Snitch	Quiet
Suspect 1	Snitch	1,1	3,0
	Quiet	0,3	2,2

The numbers in each box are players payoffs with player 1's listed first.

So what can we learn from this? Well, the Prisoner's dilemma is a situation in which there are gains from cooperation. In the Prisoner's dilemma the main issue is whether the players will cooperate (choose quiet). The issue is, though, there is an opportunity to free ride. Think about it from your opponent's perspective - she knows if she's quiet you both should get shorter sentences if you cooperate - however, doesn't that create somewhat of an incentive to free ride and to tell on the other prisoner since you'll get your optimal outcome?

However, you're risking a worse outcome than if you didn't cooperate here.

Exercise: Take a minute to work out the Nash Equilibrium for this example. Note that there may be *multiple* equilibria, or even no equilibria.

Now let's have a watch of this Youtube video to summarise what we've learned and discover the Nash Equilibrium for this example.

Let's have a browse of the Wikipedia page for this example, as it appeared on 15/06/2017, and take a look at the Generalized Prisoner's Dilemma, the Iterated Prisoner's Dilemma and some examples that arise naturally in life.

2.4.2 The Battle of The Saxes

In this example, the players agree it is best to *cooperate*, but the players **disagree** about what their preferred outcome is.

Example. - The Battle of The Saxes

Two people go out together. They have a choice of two concerts - either Bach or Stravinsky. One prefers Bach, the other Stravinsky. However they are even sadder if they don't go together.

Suppose for player 1 we have the following (abbreviated) payoff function:

$$u_1(B,B) = 2 > u_1(S,S) = 1 > u_1(B,S) = u_1(S,B) = 0.$$

Player 2 has a similarly defined payoff function.

We obtain the following table:

		Person 2	
		Bach	Stravinsky
Person 1	Bach	2,1	0,0
	Stravinsky	0,0	1,2

Rather clearly here what is happening is that both players agree to cooperate, but disagree about the best outcome. There are a number of possible applications of this - most obvious is probably politicians making decisions within a party and wanting to keep whip.

Exercise: Take a minute to work out the Nash Equilibrium for this example. Note that there may be *multiple* equilibria, or even no equilibria.

Aside: There are a few amendments to this game worth looking at, namely the one which minor changes are made to allow for the fact person 1 would probably still get some utility[7] from going to Bach, even if it was without person 2, compared to if they both went to the concert they dislike. Similarly Wikipedia has an interesting example which adds a 'burning money dimension' - worth checking out later on.

2.4.3 Matching Pennies Game

This is purely a conflict game. The rules are that two players decide to simultaneously show the head or tail of a coin. If they show the same, player 2 pays player 1 one euro, otherwise the opposite happens.

This leads to the following payoff matrix:

		Person 2	
		Head	Tail
Person 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

Exercise: Take a minute to work out the Nash Equilibrium for this example. Note that there may be *multiple* equilibria, or even no equilibria.

[7] "total satisfaction received from consuming a good or service".

Let me now introduce something that seems at odds with what we have just figured out:

Theorem 2.1. *Nash's Theorem.* *For every finite game G there exists at least one Nash equilibrium for G.*

But here there is no *pure strategy equilibrium*, where a *pure strategy* provides a complete description of how a player will play a game. In particular, it determines the move a player will make for **any situation** she could face. In our examples, we couldn't decide upon a pure strategy (choosing either heads or tails) that lead to a Nash equilibrium.

However there is **another** type of equilibrium, known as a *mixed strategy Nash equilibrium*. By Nash's Theorem, if no equilibrium exists from *pure strategies*, one must exist from *mixed strategies*.

Definition. A *mixed strategy* is a probability distribution over two or more pure strategies.

- The players then choose **randomly** amongst their options in equilibrium.
- **NB:** In equilibrium, each player's probability distribution makes all other players **indifferent** between their pure strategies.[8]

This can be a little hard to grasp at first, so let's watch this Youtube video to revise some of these concepts and see how the Nash equilibrium in mixed strategies is obtained.

...So each player chooses H or T with equal probability, which makes the other player **indifferent** to choosing H or T themselves, meaning neither has an incentive to try another strategy. This fits the definition of a Nash equilibrium.

Exercise: There is an algorithm available for determining the mixed strategy Nash equilibrium for a 2×2 game.

- Watch this Youtube video to get the basics of the algorithm.

[8] For 2×2 games like these.

- Have a read of this Handout for a more in depth example with more maths involved (read up to but not including "Hints for Finding the Mixed Nash Equilibria in Larger Games").

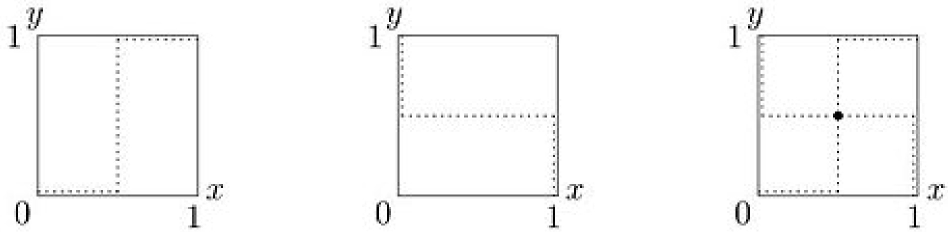


Figure 2: Mixed Strategy Nash Equilibirum for matching pennies

2.4.4 Amended Battle of the Saxes

Similar to the battle of the saxes, but now both want to go and see Bach. It seems there should be coordination, but since choice is made simultaneously this confuses things.

		Person 2	
		Bach	Stravinsky
Person 1	Bach	2,2	0,0
	Stravinsky	0,0	1,1

It seems that there should be no doubting the outcome, but this does not hold always!

Exercise: Take a minute to work out the Nash Equilibrium for this example.
Hint: What about a Nash equilibrium in mixed strategies?

2.5 A Game of Hex

We will now discuss some of the mathematics behind he game 'Hex', invented independently by mathematicians Piet Hein and John Nash. The game is traditionally played on an 11×11 rhombus composed of hexagonal compartments. Players shade in one hexagonal square per turn (no skipping turns) and the winner is the first player who manages to connect their opposite sides in an unbroken chain. See figure 3 for a game of Hex won by the blue player.

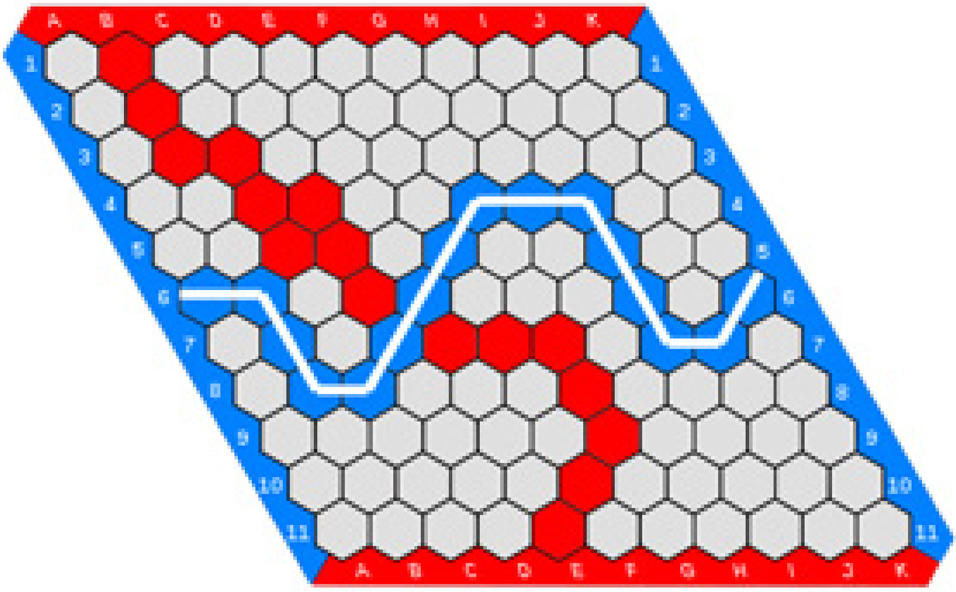
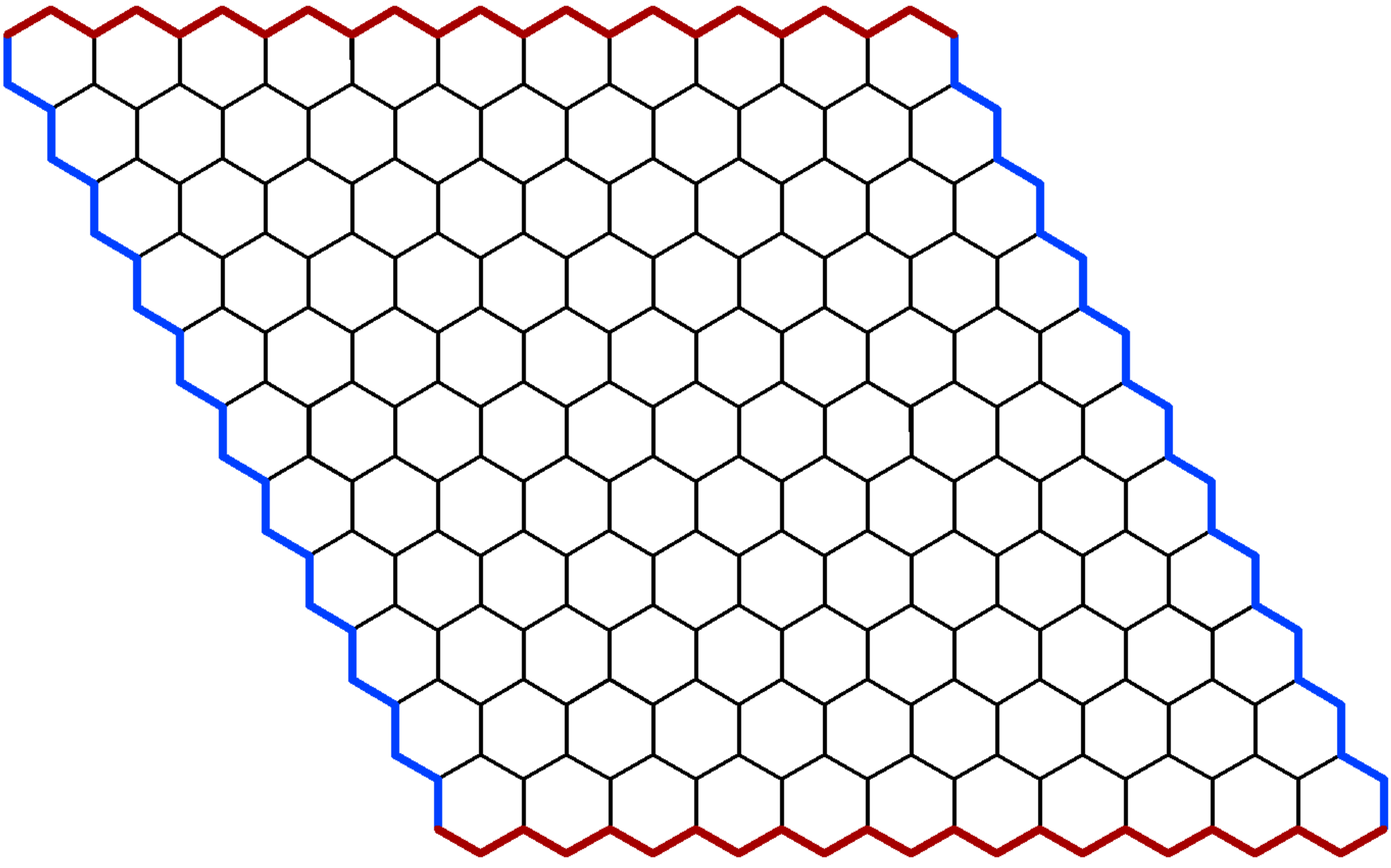


Figure 3: 11×11 Hex gameboard showing a winning configuration for Blue.

Without knowing anything about strategy or proofs, I want the class to split into random pairs of students and play against each other using the board on the next page. Use different colour pens when doing so. The winner in each pair gets a small prize from me. Remember, in game theory you are supposed to be *rational, intelligent players*, so play to the best of your ability, think your moves through and play to win!



Now let's talk about some results regarding this game:

- Hex is a *determined game* - that is, it can *never* end in a tie. There is always going to be a winner and loser. This was proven by Nash c. 1949 in the "Hex Theorem" which we will reprove ourselves shortly. Fun fact: this result is equivalent to a result known as the "Brower Fixed Point Theorem" in topology.

- In Hex, having an extra piece on the board is always an advantage, never a handicap - even if the piece is placed randomly on the board. We'll prove this result shortly too.

- The first player (first person to make a move) has a winning strategy. That is, there exists a strategy the first player can play that guarantees them to win. This was prove by Nash in 1952 and we will reprove it ourselves shortly. Note that Nash's proof was not a constructive one - he proved a winning strategy **existed**, but we don't automatically known **what that strategy is**.

- Related to the previous point, in 2002 the first explicit winning strategy on a 7×7 board was described. In the 2000s, by using brute force search computer algorithms, Hex boards up to size 9×9 (as of 2016) have been completely solved. The 11×11 board is (as of right now) still unsolved. This means you should never face an opponent on a 9×9 board or smaller - they might know the winning strategy from the get go!

- There is a "Hex Uniqueness Theorem" that says it is impossible for any Hex Board to be coloured in such a way as to satisfy winning conditions for more than one player.

- Hex is a finite, perfect information game. 'Perfect Information' means all players have all the knowledge possible of all the previous moves, all the moves that they and their opponent could make, and all of the possible consequences of any move. Chess is another example of a game with perfect information as each player can see all of the pieces on the board at all times.

- According to the Wikipedia page on Hex, in 11×11 Hex there are approximately 2.4×10^{56} possible legal positions; this compares to 4.6×10^{46} legal positions in chess.

Let's tackle the third bullet point first.

Theorem 2.2. *The first player in Hex on a board of any size has a winning strategy.*

This is a *reductio ad absurdum* existence proof attributed to John Nash. Such a proof gives no indication of a correct strategy for play. The proof is common to a number of games including Hex, and has come to be called the "strategy-stealing" argument. Here is a highly condensed informal statement of the proof:

Proof.

1. Either the first or second player must win (by the first bullet point), therefore there must be a winning strategy for either the first or second player.
2. Let us assume that the second player has a winning strategy (proof by contradiction, remember).
3. The first player can now adopt the following defence: She makes an arbitrary move. By bullet point # 2, she is not at a disadvantage because of this. Thereafter she plays the *winning second player strategy assumed above.* If in playing this strategy, she is required to play on the cell where an arbitrary move was made, she makes another arbitrary move. In this way she plays the winning strategy with one extra piece always on the board (which, again, is not a handicap). Moral of the story: the first player 'becomes' the second player.
4. Therefore the first player can win.
5. Because we have now contradicted our assumption that there is a winning strategy for the second player, we are forced to drop this assumption.
6. Consequently, there must be a winning strategy for the first player, otherwise we reach the above contradiction.

■

Comments on this:

- What is keeping player B from stealing the strategy back? As in, what if B plays another bogus move, and we're back at A being the effective first player? We are assuming perfect play, so if A's winning strategy beats B when B is playing the best she can, that strategy will also beat B when she is playing less than perfectly.

- Given this, we would then wonder, how can A steal the strategy in the first place? Player B had the winning strategy, so if A started out playing a random move, then surely B can just keep playing the winning strategy and beat A, right? This is the heart of the contradiction and our proof. Given that A played a random move and has effectively made B the first player, B is no longer in the position to use the winning strategy even though our assumption would imply that B can just keep playing perfectly and beat A. Hence, our assumption that B had the winning strategy must have been flawed.

- This strategy-stealing argument can be applied to any other symmetric game where having an extra move or game piece on the board can never hurt you. An example would be tic-tac-toe, which we played in the first few days and had a nice picture to indicate to us the best strategy for player one.

Now, the second bullet point:

Theorem 2.3. *Having extra pieces of your own colour lying on the board cannot hurt you.*

Proof. A short and sweet argument is as follows: Suppose that there is an extra piece at position x on the board. If x is part of your strategy, then on the turn when you should be playing at position x, you could instead lay down another piece somewhere else. If there is nowhere else to place your piece, the board is full and the game ended at the previous player's turn. If x is *not* part of your strategy, then you would not care that it is occupied. In either case, your strategy is unaffected so an extra piece of your own colour on the board has not hurt you, as required. ■

Finally, the first bullet point. To prove the *Hex Theorem* we will first need to learn some graph theory on the fly. Here and here are resources that are useful if you've seen graph theory before and need a refresher.

Lemma 2.4. *A finite graph whose vertices have degree at most two is the union of disjoint subgraphs, each of which is either (i) an isolated node, (ii) a simple cycle*[9]*, (iii) a simple path*[10]*.*

[9] A closed walk with unique vertices.

[10] A walk passing through each vertex once.

Proof.
We induct on the number of edges in a graph. Consider a graph g with N nodes. Each node can have degree at most two, so g can have at most N edges. For simplicity, we denote a graph with k edges as g_k. In the base case, g_0, all the nodes are isolated. When a graph has $n+1$ edges, we randomly choose an edge to remove; call the edge (u, v). The the nodes u and v now have degree at most 1 since they had degree at most two before we removed edge (u, v). Therefore u and v cannot be on any cycles. By assumption, g_n is the union of disjoint isolated nodes, simple cycles, and simple paths. We now add (u, v) back into the graph. The subgraphs that were disjoint from u and v in g_n are unchanged by the addition of (u, v), and the nodes u and v are now either on the same simple path or cycle. Therefore, g_{n+1} is also the union of disjoint isolated nodes, simple cycles, and simple paths. Hence the lemma is true for *all* g_k with $0 \leq k \leq N$; in particular this is true for g_N, as required. ■

Take some time to go back over and try understand what is going on in this proof. This is the hardest (and one of the only) proof in this course!

How is this lemma useful?

Next step: For simplicity, we substitute the coloured tiles in the game with x's and o's. We represent the game board as a graph $G = (V, E)$, with a set of vertices/nodes V and a set of edges E. **Each vertex of a hexagonal board space**[11] is a node in V, and each side of a hexagonal board space is an edge in E. We create four additional nodes, one connected to each of the four corners of the core graph; call these new nodes u_1, u_2, u_3, and u_4 and the edges that connect them to the core graph e_1, e_2, e_3, and e_4. An X-face is either a tile marked with an x or one of the regions marked X or X'. Similarly, O-face is either a tile marked with an o or one of the regions marked O or O'. Hence, the edges e_1, e_2, e_3, and e_4 lie between an X-face and an O-face since the regions X, X', O', and O are considered 'faces' as well. Let's draw a picture to illustrate this:

[11] a corner of the hexagonal tile

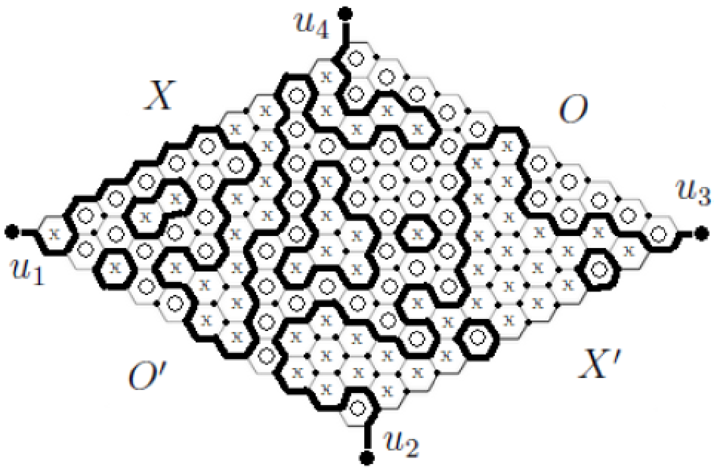


Figure 4: Hex graph.

Theorem 2.5. ***(Hex Theorem).*** *If every tile of the Hex board is marked either x or o, then there is either an x-path connecting regions X and X' or an o-path connecting regions O and O'.*

Proof.
First, we construct a subgraph $G' = (V, E')$ of G, with the same nodes but a subset of the edges. We define an edge to belong to E' only if it lies between a X-face and an O-face (figure 4). Therefore, e_1, e_2, e_3, and e_4 belong to E'. Note the nodes u_1, u_2, u_3, and u_4 each have degree one.

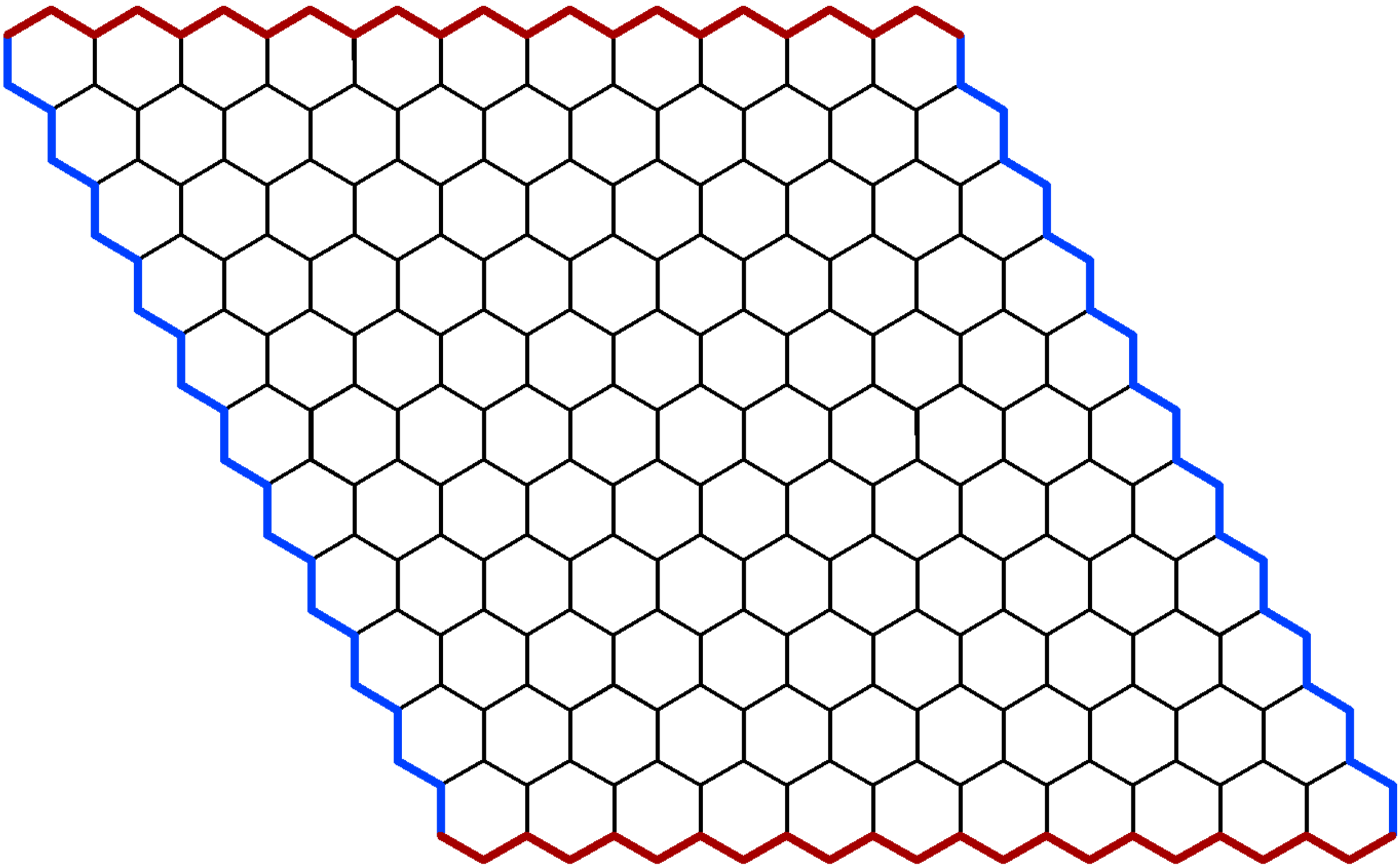
If all three hexagons around a node are marked the same, then the node is isolated in G' and has degree zero. If a node is surrounded by two hexagons of one pattern and one hexagon of the other pattern, then that node has two incident edges. Hence, each node in the core graph has degree either zero or two. Since G' has nodes with degree at most two, by the lemma, G' is a union of disjoint subgraphs, each of which are isolated nodes, simple cycles, or simple paths. Each of the nodes u_1, u_2, u_3, and u_4 are ends of some *path* because they have degree one. The disjointness of subgraphs in G' ensures that these paths do not cycle (loop back to the starting node). Therefore, there exist two simple paths in G', each connecting two of u_1, u_2, u_3, and u_4. Note that these paths cannot overlap as at an 'overlap' node, its degree would be 4, a contradiction to what we proved at the start of this paragraph.

Although the winner depends on the orientation of the paths, the paths do trace out a winning chain of hexagons. Therefore, for any arbitrary configuration of the Hex board, a winning path for one of the players exists, as required. ∎

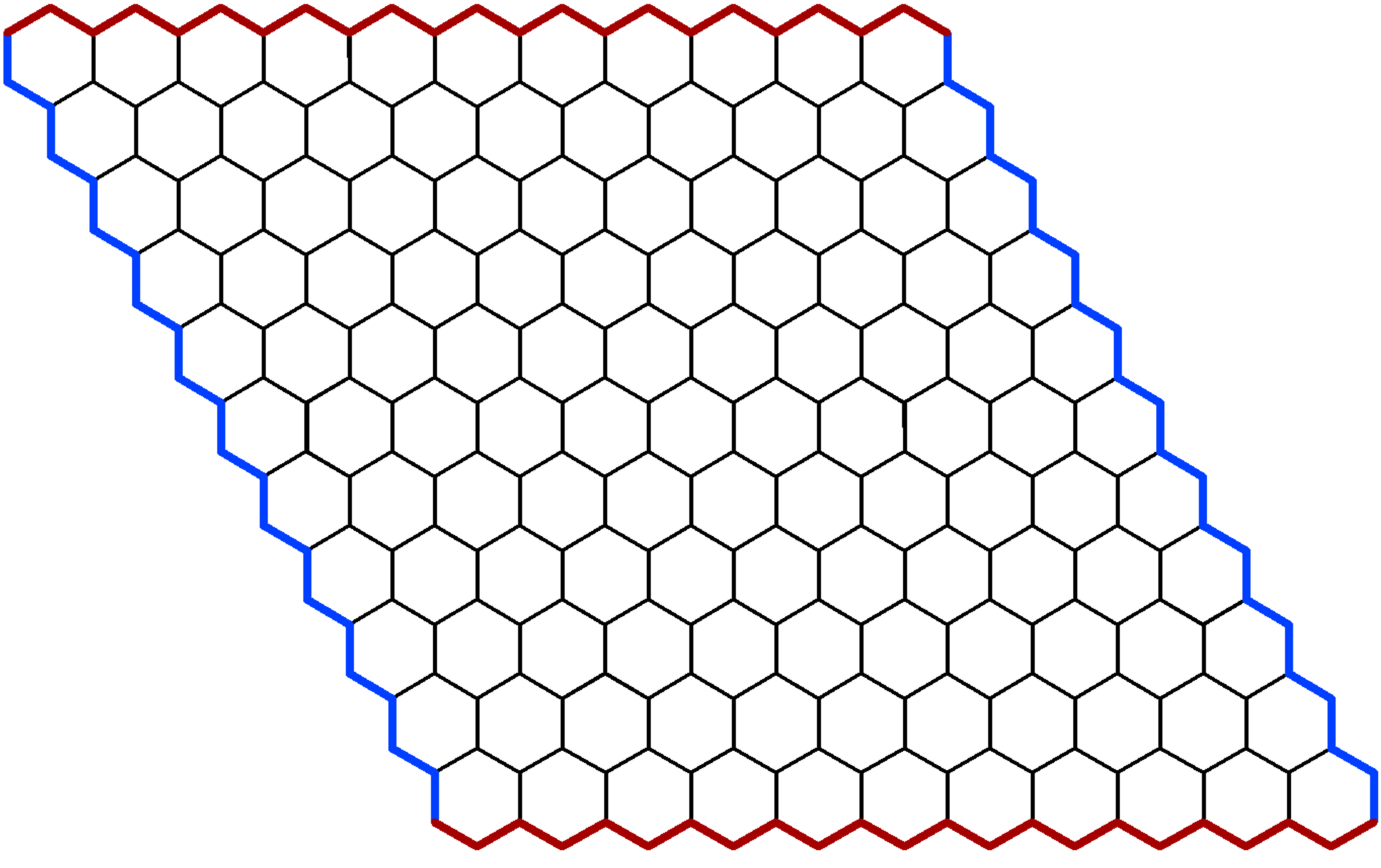
I have some notes regarding the equivalence of the Hex Theorem and the Brower Fixed Point Theorem, however we probably won't get a chance to cover the proof in class as we would need to cover a lot of topology first.

With regards to the fifth bullet point, the Hex Uniqueness Theorem, I invite you to have a read of the following paper which we will discuss the next day if we have the time: An Inductive proof of Hex Uniqueness. (You don't have to read it tonight if you don't want to - I'd rather you do the following exercise:)

Exercise: Go off tonight and google 'hex strategy'. Read up on Hex strategies (e.g. this) and we will play again tomorrow morning. I will also invite students up to the board before and after a game to explain their strategies and what did and didn't work.



A spare board:



2.6 Dominated Actions

A key thing when predicting the outcome of a game is that we wish to assume that players do not know much about their opponents optimum strategies. It's not always possible to get a solution in this way, but when we can, the solution is pretty strong. We ask the key question - how should one *not play the game* - as well as how you should play it.

		Person 2	
		Left	Right
Person 1	Up	3,0	0,-4
	Down	2,4	-1,8

Example. If you were player 1 here, what would you logically do? Well, say player 2 picks Left - we are clearly better off picking Up since $3 > 2$. Similarly, if she picks Right, it is only logical that player 1 goes for up again, as $0 > -1$. Therefore player 1 should always pick up.

Since player 2 knows player 1 will logically always pick Up, her best bet is to pick left, since $0 > -4$, as picking Right would clearly be suboptimal.

Definition. In any game we say a player's action **strictly dominates** another action if it is superior no matter what the other players do.

A strictly dominated action ('an action that is strictly dominated by another action') can never be optimal - we should never pick this if we are rational.

Some examples of this are;

- The Prisoner's dilemma - Snitch dominates Quiet.
- In the Battle of the Saxes and Matching Pennies neither strategy strictly dominates.

Definition. Let a be an action profile, in which the action of each player i is a_i. Let a'_i be any action of player i either equal to a_i or different. Then (a'_i, a_{-i}) denotes the action profile in which every player j except i chooses her action a_j as specified by a, while player i chooses a'_i.

Example. An action profile for n players could be $a = (a_1, \cdots, a_n)$. Then $(a'_i, a_{-i}) = (a_1, a_2, \cdots, a'_i, \cdots, a_n)$.

Definition. In a strategic game with ordinal preferences, player $i's$ action a_i'' *strictly dominates* her action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$$

for every list a_{-i} of the other players' actions, where u_i is a payoff function that represents player $i's$ preferences. Here, action $a_i^{'}$ is *strictly dominated.*

We also have weak domination:

Definition. In a strategic game with ordinal preferences, player $i's$ action a_i'' *weakly dominates* her action a_i' if:

$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i})$ for *every* list a_{-i} of the other players actions and $u_i(a_i'', a_{-i}) > u_i(a_i^{'}, a_{-1})$ for *some* list a_{-i} of the other players actions,

where u_i is a payoff function that represents player $i's$ preferences. Here, action a_i' is *weakly dominated.*

Example. A 'best response function' example:

Person 2

Person 1

	L	R
T	1	0
M	2	0
B	2	1

We only list player 1's payoffs for convenience.

- M weakly dominates T. If Player 2 chooses L its clearly preferred, if player 2 chooses R, we are indifferent between them $0 = 0$
- B weakly dominates M - rather clearly $2 = 2$ if 2 chooses L, but clearly $1 > 0$ if 2 chooses R
- Furthermore, B strictly dominates T. No matter what 2 does, B is *always* better than T.

It is notable that it is not always the case that we have a dominan**t** action. We can, however have a dominat**ed** action independently. It may be possible to simplify the analysis by ruling out actions that are unattractive to the corresponding player.

Example. Are there any actions in particular that you can say player 1 or 2 will never play in this?

Person 2

Person 1

	L	M	R
U	3,0	0,-5	0,-4
C	1,-1	3,3	-2,4
D	2,4	4,1	-1,8

We can clearly rule out M and C strategies - since these are clearly inferior in any scenario. From player 1's perspective it is worse than D if 2 plays M or R, and worse than U if 2 plays L or R. Similar logic holds for player 2's elimination of M. We can now picture this as a 2×2 rather than a 3×3 situation:

Person 2

Person 1

	L	R
U	3,0	0,-4
D	2,4	-1,8

Then when we further compare, U becomes the dominate action for player 1:

Suppose player 2 chooses L - we always will choose U, since $3 > 2$. Similarly, if player 2 chooses R, we choose U, since $0 > -1$. Since player 1 will always choose U, player 2 should choose L as $0 > -4$. Thus, the strategy should always be $\{U, L\}$.

Example. Now we turn to *Goldenballs*. We now watch this video to explain the rules of Goldenballs. Now suppose our grand prize is $£13,600$.

Person 2

Person 1

	Split	Steal
Split	6800,6800	0,13600
Steal	13600,0	0,0

The Steal strategy weakly dominates Split here rather clearly - from players 1's perspective if player 2 steals then she's indifferent no matter what she does, but if player 2 splits, she should rather clearly steal. Hence the weak domination. Note that morality does not factor into this equation, unlike the 'Joker & his boatload of fun' scenario.

The fun thing can be seen in what this guy does. He says he will steal - but will split the money after the show. Whats interesting here is he effectively *changed* the payoff matrix - he changed the dominant strategy and the matrix to this:

		Person 2	
		Split	Steal
Person 1	Is Honest	6800,6800	0,0
	Dishonest	13600,0	0,0

Split now dominates Steal for player 2 - he changed the perspective of the game. Look at it like this: why would player 2 have any incentive to pick steal? Instead, both players ended up splitting anyway. Player 1 rather cleverly beat the game.

Aside: Some fun papers have been done on the game theory aspects to Goldenballs, most notably one by Thaler et al. in their paper *Split or Steal? Cooperative Behaviour when the stakes are large?*. Worth at a quick read if you have the time.

2.7 Nash Equilibrium

NB: Many games cant be solved by iteritavely eliminating dominated actions, so in general we need a more general equilibrium concept. Again, instead of asking what the solution of a given game is we instead ask what outcomes cannot be the solution. For a given outcome to be a solution it must be the case that players are willing to choose the actions the theory predicts they will play. Therefore, any outcome where an individual player could receive a higher payoff from playing a different action cannot be an equilibrium.

In other terms, we basically ask what *cannot be a solution* - and we need to rule out any action where the player could get a higher payoff from another solution.

The concept of a Nash equilibrium is thought of via the following key questions:

- What is the outcome of a game?
- What actions will be chosen by the players in a strategic game?

- Rationally describing our best outcome, however, what really the strategy leading to our best outcome?
- Our best outcome is fully dependent on the choices of other players. Each player needs to form a belief about what they *think* other people will choose.

Definition. A Nash Equilibrium is an action profile a^* with the property that no player i can do better by choosing an action different from a_i^* given that every other player j adheres to a_j^*.

No one player has an incentive to deviate from a^*.

In a setting in which all the players in any given play of the game are drawn randomly from some population, a Nash Equilibrium corresponds to a *steady state*. A Nash equilibrium embodies a *stable social norm* - if everyone else adheres to it why would anyone want to deviate from it?

The final, formalised definition of a *Nash Equilibrium* is thus:

Definition. The action profile a^* in a strategic game with ordinal preferences is a *Nash Equilibrium* if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a_{-i}^*) in which player i chooses a_i while every other player j chooses a_j^*.

This is equivalent to state that for every player i,

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*).$$

This definition implies neither that a game necessarily has a Nash equilibrium nor that it has at most one. However, recalling Nash's Theorem, every game (that we are interested in) does indeed have at least one Nash Equilibrium. For pure strategies we may not have a Nash Equilibrium, so we can turn to mixed strategies.

We can find the Nash equilibria of a game in which each player only has a few actions by examining each action profile in turn to see if it satisfies the conditions for equilibrium. In more complicated games, however, it is often better to work with the players best response functions.

2.8 Best response functions

Definition. Define a *weak best response function*[12] B_i for player i as:

$$B_i(a_{-i}) := \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i\}$$

where A_i is the action profile for player i and a_{-i} is the list of other player's actions.

The concept: Any action in $B_i(a_{-i})$ is at least as good for player i as every other action of player i when the other players actions are given by a_{-i}.

It is simply our best response - the strategies which give the highest payoffs.

Sometimes B_i can be a singular set - in which case it would be a *strong best response function.* In other cases, B_i would include more than one value, hence it would be a weak best response function.

We can connect this to Nash equilibria: *A Nash Equilibrium is an action profile for which every player's action is a best response to every other players' actions.*

Example.

Person 2

Person 1		L	C	R
	T	1,2	2,1	1,0
	M	2,1	1,1	0,0
	B	0,1	0,0	1,2

Step 1: What are player 1's best responses?

- If player 2 picks L, she (player 1) should pick M. So $B_1(L) = \{M\}$.
- If player 2 picks C, she should pick T. So $B_1(C) = \{T\}$.
- If player 2 picks R, she should pick T or B. So $B_1(R) = \{T, B\}$

Step 2: What are player 2's best responses?

[12] This isn't a 'function' in the usual sense as there is not a unique output.

- If player 1 picks T, she (player 2) should pick L. So $B_2(T) = \{L\}$.
- If player 1 picks M, she should pick either L or C. So $B_2(M) = \{L, C\}$.
- If player 1 picks B, she should pick R. So $B_2(B) = \{R\}$.

Represent this as a table, with $*'s$ beside the best responses:

Person 2

Person 1		L	C	R
	T	1,2*	2*,1	1*,0
	M	2*,1*	1,1*	0,0
	B	0,1	0,0	1*,2*

We have two situations where these best functions intersect: $\{M, L\}$ and $\{B, R\}$. These are hence our Nash equilibria, as required.

2.9 Pandemic!

Exercise: Next class we are going to play the board game Pandemic!. Please watch this Youtube video on how to set up the game and the moves you can make. Pandemic! is a somewhat complicated game to play if you've never seen it before, so try watch the above video two or three times to understand the rules before we start. Here is a picture of the role cards available to you. We will go through the rules again before we play tomorrow.

IMPORTANT: Do **not** google or find out elsewhere any sort of strategies to play. I want all new players to the game to figure out what strategies work best **for themselves**. Tomorrow I will distribute students who have played the game before ('veterans') amongst the boards.

2.10 Gameshows and Game Theory - Simultaneous Move Games in Action

We have seen the *Goldenballs Example*, however what else can we do from here? We can fortunately see some of these games in real life and many scenarios from media. Let's try and model them!

2.10.1 The Princess Bride - Poison Scene

Now for something a tad different:

Let's analyse the text from this scene: (Source)

Our hero Westley, in the guise of the Dread Pirate Roberts, confronts his foe-for-the-moment, the Sicilian, Vizzini. Westley challenges him to a *Battle of Wits*; two glasses are placed on the table, each containing wine and one purportedly containing poison. The challenge, simply, is to select the glass that does not lead to immediate death.

Roberts: All right: where is the poison? The battle of wits has begun. It ends when you decide and we both drink, and find out who is right and who is dead.

Vizzini: But it's so simple. All I have to do is divine from what I know of you. Are you the sort of man who would put the poison into his own goblet, or his enemy's? Now, a clever man would put the poison into his own goblet, because he would know that only a great fool would reach for what he was given. I'm not a great fool, so I can clearly not choose the wine in front of you. But you must have known I was not a great fool; you would have counted on it, so I can clearly not choose the wine in front of me.

Roberts: You've made your decision then?

Vizzini: Not remotely. Because iocane comes from Australia, as everyone knows. And Australia is entirely peopled with criminals. And criminals are used to having people not trust them, as you are not trusted by me. So I can clearly not choose the wine in front of you.

Roberts: Truly, you have a dizzying intellect.

[The scene, beyond providing some comic relief on the theme of common knowledge, also has an important lesson on strategic moves; if the rules of the game may be changed, then the game can be rigged to one player's advantage:]

Vizzini: Let's drink – me from my glass, and you from yours.

[Allowing Roberts to drink first, he swallows his wine]

Roberts: You guessed wrong.

Vizzini (roaring with laughter): You only think I guessed wrong – that's what's so funny! I switched glasses when your back was turned. You fool.

You fell victim to one of the classic blunders. The most famous is "Never get involved in a land war in Asia." But only slightly less well known is this: "Never go in against a Sicilian when death is on the line."

[He laughs and roars and cackles and whoops until he falls over dead.]

Later on in the scene:

Buttercup: To think – all that time it was your cup that was poisoned. Roberts: They were both poisoned. *I spent the last few years building up an immunity to iocane powder.*

The movie contains several other scenes with game-theoretic themes, including many on bluffing.

So how do we model this?

In order to make movies both complex and interesting many screenwriters employ game theory unknowingly within their plots. One classic example of this is in the 1987 version of the movie "The Princess Bride", as we saw above. To summerize: in this movie there is a scene known as the *Battle of Wits* in which the main character Westley challenges a cunning man known as Vizzini in a game for a Princess's life. In this game Westley takes two goblets of wine behind his back and tells Vizzini that he has put iocane powder, a deadly poison, into only one of the goblets. He then instructs Vizinni to pick the goblet he wishes to drink and the man left standing wins the Princess. In this setup of the game, the payoff matrix is as follows where A is Westley and B is Vizzini:

Vizzini's situation - poison in cup A:

		Person B	
		A's cup	B's cup
Person A	A's cup	n/a	-1,1
	B's cup	1, -1	n/a

Vizzini's situation - poison in cup B:

		Person B	
		A's cup	B's cup
Person A	A's cup	n/a	1,-1
	B's cup	-1, 1	n/a

Vizzini does not know what goblet the poison is in, therefore there are two payoff matrices based on the possibility that the poison is in either cup. In this scenario each player must play *opposite strategies*, meaning that both players cannot drink from the same cup. This is why the 'n/a' appears in blocks of the matrix in which both players must drink from the same cup. However, one can see from these tables that there is no dominant strategy considering that Vizzini does not know where the poison is located. He has an equal chance of living or dying regardless of what cup he picks therefore the game appears to be a game of chance rather than wits.

However, what Vizzini does not know is that Westley has spent years developing immunity to iocane powder and has placed the poison *in both cups* rather than just one. In this case, from Westley's perspective, the payoff matrix is as follows:

Westley's situation - poison in both cups (and A is immune):

		Person B	
		A's cup	B's cup
Person A	A's cup	n/a	1,-1
	B's cup	1, -1	n/a

In this case, no matter what strategy B, or Vizzini, employs he will still end up with a negative payoff, meaning that he will die.

This game pertains to what we have learned in class with a few twists. First, in class we learned about dominant strategies and Nash Equilibria. However, from Vizzini's point of view, there is no dominant strategy or Nash Equilibriums because he will change his strategy depending on whether the poison is in his cup or Westley's cup. This almost creates a 'matching pennies' situation of sorts. However, this situation is slightly different then 'matching pennies' (where mixed strategies were the key) because not all of the strategies are permitted (i.e. Player A and Player B cannot both drink from cup A). In addition, in this version there are two matrices that need to be considered rather than one. In addition, this game has a deceiving trick because both players are considering different payoff matrices since Westley has deceived Vizzini. This means that Vizzini is playing a losing game from the start, which is what makes the game both interesting and funny to the viewer.

In all, the game theory in the *Battle of Wits* proves that the battle is a losing game and truly involves no wits at all.

(Source.)

2.11 Discussion

Can we think of any real life or fictional scenarios with game theory applications? Two that I've been thinking about:

- *Liar!* Isaac Asimov & the Three Laws of Robotics (summary).
- *Star Wars:* Sith vs. Jedi in *Revenge of the Sith*. Watch.

We will discuss these in class and your homework will be to come up with others (fact or fiction, up to you).

3 Political Economy

This is one of the other major applications of the field - indeed game theory is credited in parts with solving the famous **Cuban Missile Crisis**.

3.1 Cuban Missile Crisis - Stephen J. Brahms

"We're eyeball to eyeball, and I think the other fellow just blinked" were the eerie words of Secretary of State Dean Rusk at the height of the Cuban missile crisis in October 1962. He was referring to signals by the Soviet Union that it desired to defuse the most dangerous nuclear confrontation ever to occur between the superpowers, which many analysts have interpreted as a classic instance of nuclear "Chicken".

"Chicken" is the usual game used to model conflicts in which the players are on a collision course. The players may be drivers approaching each other on a narrow road, in which each has the choice of swerving to avoid a collision or not swerving. In the novel *Rebel without a Cause*, which was later made into a movie starring James Dean, the drivers were two teenagers, but instead of bearing down on each other they both raced toward a cliff, with the object being not to be the first driver to slam on his brakes and thereby "chicken out", while, at the same time, not plunging over the cliff.

While ostensibly a game of Chicken, the Cuban missile crisis is in fact not well modelled by this game. Another game more accurately represents the preferences of American and Soviet leaders, but even for this game standard game theory does not explain their choices.

On the other hand, the "theory of moves," which is founded on game theory but radically changes its standard rules of play, does retrodict - make past predictions of - the leaders' choices. More important, the theory explicates the dynamics of play, based on the assumption that players think not just about the immediate consequences of their actions but their repercussions for future play as well.

I will use the Cuban missile crisis to illustrate parts of this theory, which is not just an abstract mathematical model but one that mirrors the real-life choices, and underlying thinking, of flesh-and-blood decision makers. Indeed, Theodore Sorensen, special counsel to President John Kennedy, used the language of "moves" to describe the deliberations of *Excom*, the Executive Committee of key advisers to Kennedy during the Cuban missile crisis:

"We discussed what the Soviet reaction would be to any possible move by the United States, what our reaction with them would have to be to that Soviet action, and so on, trying to follow each of those roads to their ultimate conclusion."

Classical Game Theory and the Cuban Missile Crisis.
A recap: Game theory is a branch of mathematics concerned with decision-making in social interactions. It applies to situations (games) where there are two or more people (called players) each attempting to choose between two more more ways of acting (called strategies). The possible outcomes of a game depend on the choices made by all players, and can be ranked in order of preference by each player.

In some two-person, two-strategy games, there are combinations of strategies for the players that are in a certain sense "stable". This will be true when neither player, by departing from its strategy, can do better. Two such strategies are together known as a Nash equilibrium, named after John Nash, a mathematician who received the Nobel prize in economics in 1994 for his work on game theory. Nash equilibria do not necessarily lead to the best outcomes for one, or even both, players. Moreover, for the games that we can analyze - in which players can only rank outcomes ("ordinal games") but not attach numerical values to them ("cardinal games") - they may not always exist. (While they always exist, as Nash showed, in cardinal games, Nash equilibria in such games may involve mixed strategies, which have been mentioned briefly.)

The Cuban missile crisis was precipitated by a Soviet attempt in October 1962 to install medium-range and intermediate-range nuclear-armed ballistic missiles in Cuba that were capable of hitting a large portion of the United States. The goal of the United States was immediate removal of the Soviet missiles, and U.S. policy makers seriously considered two strategies to achieve this end.

- A Naval Blockade - or "quarantine" as it was euphemistically called, to prevent shipment of more missiles, possibly followed by stronger action to induce the Soviet Union to withdraw the missiles already installed.
- A "Surgical Air Strike" to wipe out the missiles already installed, insofar as possible, perhaps followed by an invasion of the island.

The Soviets could choose to

- Withdraw their missiles

- Maintain their missiles

		Soviet Union (S.U.)	
		Withdrawal (W)	Maintenance (M)
United States (U.S.)	Blockade (B)	Compromise (3,3)	Soviet victory, U.S. defeat (2,4)
	Air strike (A)	U.S. victory, Soviet defeat (4,2)	Nuclear war (1,1)

Figure 5: The Missile Crisis as a Game

These strategies can be thought of as alternative courses of action that the two sides, or "players" in the parlance of game theory, can choose. They lead to four possible outcomes, which the players are assumed to rank as follows:

4=best; 3=next best; 2=next worst; and l=worst.

Thus, the higher the number, the greater the payoff; but the payoffs are only ordinal, that is, *they indicate an ordering of outcomes from best to worst, not the degree to which a player prefers one outcome over another.* The first number in the ordered pairs for each outcome is the payoff to the row player (United States), the second number the payoff to the column player (Soviet Union).

Needless to say, the strategy choices, probable outcomes, and associated payoffs shown in Figure 1 provide only a skeletal picture of the crisis as it developed over a period of thirteen days. Both sides considered more than the two alternatives listed, as well as several variations on each. The Soviets, for example, demanded withdrawal of American missiles from Turkey as a quid pro quo for withdrawal of their own missiles from Cuba, a demand publicly ignored by the United States.

Nevertheless, most observers of this crisis believe that the two superpowers were on a collision course, which is actually the title of one book describing this nuclear confrontation (*Collision Course*). They also agree that neither side was eager to take any irreversible step, such as one of the drivers in Chicken might do by defiantly ripping off the steering wheel in full view of the other driver, thereby foreclosing the option of swerving.

Although in one sense the United States 'won' by getting the Soviets to withdraw their missiles, Premier Nikita Khrushchev of the Soviet Union at the same time extracted from President Kennedy a promise not to invade Cuba, which seems to indicate that the eventual outcome was a compromise of sorts. But this is not game theory's prediction for Chicken, because the strategies associated with compromise do not constitute a Nash equilibrium.

To see this, assume play is at the compromise position (3,3), that is, the U.S. blockades Cuba and the S.U. withdraws its missiles. This strategy is not stable, because both players would have an incentive to defect to their more belligerent strategy. If the U.S. were to defect by changing its strategy to airstrike, play would move to (4,2), improving the payoff the U.S. received; if the S.U. were to defect by changing its strategy to maintenance, play would move to (2,4), giving the S.U. a payoff of 4. (This classic game theory setup gives us no information about which outcome would be chosen, because the table of payoffs is symmetric for the two players. This is a frequent problem in interpreting the results of a game theoretic analysis, where more than one equilibrium position can arise.) Finally, should the players be at the mutually worst outcome of (1,1), that is, nuclear war, both would obviously desire to move away from it, making the strategies associated with it, like those with (3,3), unstable.

The Crisis as a Game of Chicken.
Using Chicken to model a situation such as the Cuban missile crisis is problematic not only because the (3,3) compromise outcome is unstable but also because, in real life, the two sides did not choose their strategies simultaneously, or independently of each other, as assumed in the game of Chicken described above. The Soviets responded specifically to the blockade after it was imposed by the United States. Moreover, the fact that the United States held out the possibility of escalating the conflict to at least an air strike indicates that the initial blockade decision was not considered final - that is, the United States considered its strategy choices still open after imposing the blockade.

As a consequence, this game is better modelled as one of *sequential bargaining*, in which neither side made an all-or-nothing choice but rather both considered alternatives, especially should the other side fail to respond in a manner deemed appropriate. In the most serious breakdown in the nuclear deterrence relationship between the superpowers that had persisted from World War II until that point, each side was gingerly feeling its way, step by ominous step. Before the crisis, the Soviets, fearing an invasion of Cuba by the United States and also the need to bolster their international strategic position, concluded that installing the missiles was worth the risk. They thought that the United States, confronted by a fait accompli[13], would be deterred from invading Cuba and would not attempt any other severe reprisals. Even if the installation of the missiles precipitated a crisis, the Soviets did not reckon the probability of war to be high (President Kennedy estimated the chances of war to be between 1/3 and 1/2 during the crisis), thereby making it rational for them to risk provoking the United States.

There are good reasons to believe that U.S. policymakers did not view the confrontation to be Chicken-like, at least as far as they interpreted and ranked the possible outcomes. The article offers an alternative representation of the Cuban missile crisis in the form of a game I will call "Alternative", retaining the same strategies for both players as given in Chicken but presuming a different ranking and interpretation of outcomes by the United States [see Figure 2]. These rankings and interpretations fit the historical record better than those of "Chicken", as far as can be told by examining the statements made at the time by President Kennedy and the U.S. Air Force, and the type and number of nuclear weapons maintained by the S.U. (more on this below).

1. BW: The choice of blockade by the United States and withdrawal by the Soviet Union remains the compromise for both players - (3,3).
2. BM: In the face of a U.S. blockade, Soviet maintenance of their missiles leads to a Soviet victory (its best outcome) and U.S. capitulation (its worst outcome) - (1,4).
3. AM: An air strike that destroys the missiles that the Soviets were maintaining is an 'honourable' U.S. action (its best outcome) and thwarts the Soviets (their worst outcome) - (4,1).
4. AW: An air strike that destroys the missiles that the Soviets were withdrawing is a 'dishonorable' U.S. action (its next-worst outcome) and thwarts the Soviets (their next-worst outcome) - (2,2).

[13] A thing that has already happened or been decided before those affected hear about it, leaving them with no option but to accept it.

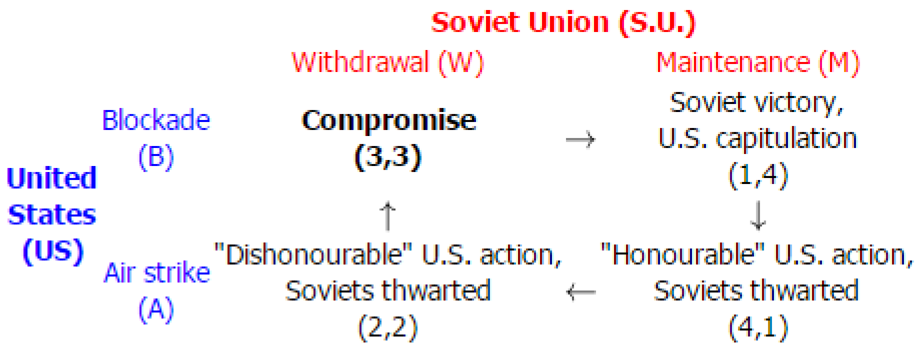


Figure 6: The Missile Crisis as a Game - Part 2

Even though an air strike thwarts the Soviets at both outcomes (2,2) and (4,1), I interpret (2,2) to be less damaging for the Soviet Union. This is because world opinion, it may be surmised, would severely condemn the air strike as a flagrant overreaction - and hence a "dishonourable" action of the United States - if there were clear evidence that the Soviets were in the process of withdrawing their missiles anyway. On the other hand, given no such evidence, a U.S. air strike, perhaps followed by an invasion, would action to dislodge the Soviet missiles.

The statements of U.S. policy makers support Alternative. In responding to a letter from Khrushchev, Kennedy said,

"If you would agree to remove these weapons systems from Cuba . . . we, on our part, would agree . . . (a) to remove promptly the quarantine measures now in effect and (b) to give assurances against an invasion of Cuba,"

which is consistent with Alternative since (3,3) is preferred to (2,2) by the United States, whereas (4,2) is not preferred to (3,3) in Chicken. If the Soviets maintained their missiles, the United States preferred an air strike to the blockade. As Robert Kennedy, a close adviser to his brother during the crisis, said,

"If they did not remove those bases, we would remove them,"

which is consistent with Alternative, since the United States prefers (4,1) to (1,4) but not (1,1) to (2,4) in Chicken. Finally, it is well known that several of President Kennedy's advisers felt very reluctant about initiating an attack against Cuba without exhausting less belligerent courses of action that might bring about the removal of the missiles with less risk and greater sensitivity to American ideals and values. Pointedly, Robert Kennedy claimed that an immediate attack would be looked upon as "a Pearl Harbor in reverse, and it would blacken the name of the United States in the pages of history," which is again consistent with the Alternative since the United States ranks AW next worst (2) - a 'dishonourable' U.S. action - rather than best (4) - a U.S. victory - in Chicken.

If Alternative provides a more realistic representation of the participants' perceptions than Chicken does, standard game theory offers little help in explaining how the (3,3) compromise was achieved and rendered stable. As in Chicken, the strategies associated with this outcome are not a Nash equilibrium, because the Soviets have an immediate incentive to move from (3,3) to (1,4).

However, unlike Chicken, **Alternative has no outcome at all that is a Nash equilibrium, except in mixed strategies.** Recall: these are strategies in which players randomize their choices, choosing each of their two so-called pure strategies with specified probabilities. But mixed strategies cannot be used to analyse Alternative, because to carry out such an analysis, there would need to be numerical payoffs assigned to each of the outcomes, not the rankings I have assumed.

The instability of outcomes in Alternative can most easily be seen by examining the cycle of preferences, indicated by the arrows going in a clockwise direction in this game. Following these arrows shows that this game is cyclic, with one player always having an immediate incentive to depart from every state: the Soviets from (3,3) to (1,4); the United States from (1,4) to (4,1); the Soviets from (4,1) to (2,2); and the United States from (2,2) to (3,3). Again we have indeterminacy, but not because of multiple Nash equilibria, as in Chicken, but rather because there are no equilibria in pure strategies in Alternative.

Rules of Play in The Game.
How, then, can we explain the choice of (3,3) in Alternative, or Chicken for that matter, given its nonequilibrium status according to standard game theory? It turns out that (3,3) is a "nonmyopic equilibrium" in both games, and uniquely so in Alternative, according to the theory of moves (**TOM**). **By postulating that players think ahead not just to the immediate consequences of making moves, but also to the consequences of countermoves to these moves, counter-countermoves, and so on, TOM extends the strategic analysis of conflict into the more distant future.**

To be sure, game theory allows for this kind of thinking through the analysis of “game trees”, where the sequential choices of players over time are described. But the game tree continually changed with each development in the crisis. By contrast, what remained more or less constant was the configuration of payoffs of Alternative, though where the players were in the matrix changed. In effect, TOM, by describing the payoffs in a single game but allowing players to make successive calculations of moves to different positions within it, adds nonmyopic thinking to the economy of description offered by classical game theory.

The founders of game theory, John von Neumann and Oskar Morgenstern, defined a game to be “the totality of rules of play which describe it.” While the rules of TOM apply to all games between two players, here I will assume that the players each have just two strategies. The four rules of play of TOM describe the possible choices of the players at each stage of play:

Rules of Play:

1. Play starts at an initial state, given at the intersection of the row and column of a payoff matrix.
2. Either player can unilaterally switch their strategy, or make a move, and thereby change the initial state into a new state, in the same row or column as the initial state. The player who switches is called player l (P1).
3. Player 2 (P2) can respond by unilaterally switching their strategy, thereby moving the game to a new state.
4. The alternating responses continue until the player (P1 or P2) whose turn it is to move next chooses *not* to switch their strategy. When this happens, the game terminates in a final state, which is the outcome of the game.
5. A player will not move from an initial state if this move (i) leads to a less preferred outcome, or (ii) returns play to the initial state, making this state the outcome.
6. If it is rational for one player to move and the other player not to move from the initial state, the move takes precedence: it overrides staying, so the outcome will be induced by the player that moves.

Note that the sequence of moves and countermoves is strictly alternating: first, say, the row player moves, then the column player, and so on, until one player stops, at which point the state reached is final and, therefore, the outcome of the game. I assume that no payoffs accrue to players from being in a state unless it becomes the outcome (which could be the initial state if the players choose not to move from it).

To assume otherwise would require that payoffs be numerical, rather than ordinal ranks, which players can accumulate as they pass through states. But in many real-life games, payoffs cannot easily be quantified and summed across the states visited. Moreover, the big reward in many games depends overwhelmingly on the final state reached, not on how it was reached. In politics, for example, the payoff for most politicians is not in campaigning, which is arduous and costly, but in *winning*.

Rule l differs drastically from the corresponding rule of play in standard game theory, in which players simultaneously choose strategies in a matrix game that determines its outcome. Instead of starting with strategy choices, TOM assumes that players are already in some state at the start of play and receive payoffs from this state only if they stay. Based on these payoffs, they must decide, individually, whether or not to change this state in order to try to do better.

Of course, some decisions are made collectively by players, in which case it is reasonable to say that they choose strategies from scratch, either simultaneously or by coordinating their choices. But if, say, two countries are coordinating their choices, as when they agree to sign a treaty, the important strategic question is what individualistic calculations led them to this point. The formality of jointly signing the treaty is the culmination of their negotiations and does not reveal the move-counter-move process that preceded the signing. **NB:** It is precisely these negotiations, and the calculations underlying them, that TOM is designed to uncover.

To continue this example, the parties that sign the treaty were in some prior state from which both desired to move - or, perhaps, only one desired to move and the other could not prevent this move from happening (rule 6). Eventually they may arrive at a new state, after, say, treaty negotiations, in which it is rational for both countries to sign the treaty that was previously negotiated.

As with a treaty signing, almost all outcomes of games that we observe have a history. TOM seeks to explain strategically the progression of (temporary) states that lead to a (more permanent) outcome. Consequently, play of a game starts in an initial state, at which players collect payoffs only if they remain in that state so that it becomes the final state, or outcome, of the game.

If they do not remain, they still know what payoffs they would have collected had they stayed; hence, they can make a rational calculation of the advantages of staying or moving. They move precisely because they calculate that they can do better by switching strategies, anticipating a better outcome when the move-countermove process finally comes to rest. The game is different, but not the configuration of payoffs, when play starts in a different state.

Rules l - 4 (rules of play) say nothing about what causes a game to end, only when: termination occurs when a "player whose turn it is to move next chooses not to switch its strategy" (rule 4). But when is it rational not to continue moving, or not to move at all from the initial state?

Rule 5 (termination rule) says when a player will not move from an initial state. While condition (i) of this rule needs no defence, condition (ii) requires justification. **It says that if it is rational, after P1 moves, for play of the game to cycle back to the initial state, P1 will not move in the first place.** After all, what is the point of initiating the move-counter-move process if play simply returns to square one, given that the players receive no payoffs along the way to the outcome?

Backward Induction.
To determine where play will end up when at least one player wants to move from the initial state, I assume the players use *backward induction.* This is a reasoning process by which the players, working backward from the last possible move in a game, anticipate each other's rational choices. For this purpose, I assume that each has complete information about the other's preferences, so each can calculate the other player's rational choices, as well as its own, in deciding whether to move from the initial state or any subsequent state.

To illustrate backward induction, consider again the game Alternative in Figure 6. After the missiles were detected and the United States imposed a blockade on Cuba, the game was in state BM, which is worst for the United States (1) and best for the Soviet Union (4). Now consider the clockwise progression of moves that the United States can initiate by moving to AM, the Soviet Union to AW, and so on, assuming the players look ahead to the possibility that the game makes one complete cycle and returns to the initial state (state 1):

	State 1		State 2		State 3		State 4		State 1
	U.S.		S.U.		U.S.		S.U.		
U.S. starts	(1,4)	→	(4,1)	→	(2,2)	→ |	(3,3)	→	(1,4)
Survivor	(2,2)		(2,2)		(2,2)		(1,4)		

Figure 7: First game tree.

This is a game tree, though drawn horizontally rather than vertically. The survivor is a state selected at each stage as the result of backward induction. It is determined by working backward from where play, theoretically, can end up (state 1, at the completion of the cycle).

Assume the players' alternating moves have taken them clockwise in Alternative from (1,4) to (4,1) to (2,2) to (3, 3), at which point S.U. in state 4 must decide whether to stop at (3,3) or complete the cycle by returning to (1,4). Clearly, S.U. prefers (1,4) to (3,3), so (1,4) is listed as the survivor below (3,3): because S.U. would move the process back to (1,4) should it reach (3,3), the players know that if the move-countermove process reaches this state, the outcome will be (1,4).

Knowing this, would U.S. at the prior state, (2,2), move to (3,3)? Because U.S. prefers (2,2) to the survivor at (3,3) - namely, (1,4) - the answer is no. Hence, (2,2) becomes the survivor when U.S. must choose between stopping at (2,2) and moving to (3,3) - which, as I just showed, would become (1,4) once (3,3) is reached.

At the prior state, (4,1), S.U. would prefer moving to (2,2) than stopping at (4,1), so (2,2) again is the survivor if the process reaches (4,1). Similarly, at the initial state, (1,4), because U.S. prefers the previous survivor, (2,2), to (1,4), (2,2) is the survivor at this state as well.

The fact that (2,2) is the survivor at the initial state, (1,4), means that it is rational for U.S. to move to (4,1), and S.U. subsequently to (2,2), where the process will stop, making (2,2) the rational choice if U.S. moves first from the initial state, (1,4). That is, after working backwards from S.U.'s choice of completing the cycle or not from (3,3), the players can reverse the process and, looking forward, determine what is rational for each to do. I indicate that it is rational for the process to stop at (2,2) by the vertical line blocking the arrow emanating from (2,2), and underscoring (2,2) at this point.

Observe that (2,2) at state AM is worse for both players than (3,3) at state

BW. Can S.U., instead of letting U.S. initiate the move-countermove process at (1,4), do better by seizing the initiative and moving, counterclockwise, from its best state of (1,4)? Not only is the answer yes, but it is also in the interest of U.S. to allow S.U. to start the process, as seen in the following counterclockwise progression of moves from (1,4):

	State 1		**State 2**		**State 3**		**State 4**		**State 1**
S.U. starts	S.U. (1,4)	→	U.S. $\underline{(3,3)}$	→ |	S.U. (2,2)	→	U.S. (4,1)	→	(1,4)
Survivor	(3,3)		(3,3)		(2,2)		(4,1)		

Figure 8: Second game tree.

S.U., by acting "magnanimously" in moving from victory (4) at BM to compromise (3) at BW, makes it rational for U.S. to terminate play at (3,3), as seen by the blocked arrow emanating from state 2. This, of course, is exactly what happened in the crisis, with the threat of further escalation by the United States, including the forced surfacing of Soviet submarines as well as an air strike (the U.S. Air Force estimated it had a 90 percent chance of eliminating all the missiles), being the incentive for the Soviets to withdraw their missiles.

Applying TOM.

Like any scientific theory, TOM's calculations may not take into account the empirical realities of a situation. In the second backward-induction calculation, for example, it is hard to imagine a move by the Soviet Union from state 3 to state 4, involving maintenance (via reinstallation?) of their missiles after their withdrawal and an air strike. However, if a move to state 4, and later back to state 1, were ruled out as infeasible, the result would be the same: commencing the backward induction at state 3, it would be rational for the Soviet Union to move initially to state 2 (compromise), where play would stop.

Compromise would also be rational in the first backward-induction calculation if the same move (a return to maintenance), which in this progression is from state 4 back to state 1, were believed unfeasible: commencing the backward induction at state 4, it would be rational for the United States to escalate to air strike to induce moves that carry the players to compromise at state 4. Because it is less costly for both sides if the Soviet Union is the initiator of compromise - eliminating the need for an air strike - it is not surprising that this is what happened.

To sum up, the Theory of Moves renders game theory a more dynamic theory. By postulating that players think ahead not just to the immediate consequences of making moves, but also to the consequences of counter-moves to those moves, counter-counter-moves, and so on, it extends the strategic analysis of conflicts into the more distant future. TOM has also been used to elucidate the role that different kinds of power - moving, order and threat - may have on conflict outcomes, and to show how misinformation can affect player choices. These concepts and the analysis have been illustrated by numerous cases, ranging from conflicts in the Bible to disputes and struggles today.

(Source).

3.2 Mutually Assured Destruction

Let's look at this from another perspective.

When the atom was split, a Pandora's box was opened. This scientific advancement led to the development of the atomic bomb – humankind had never before possessed such a destructive weapon. The United States was the first to successfully develop the atomic bomb and the first to show the bomb's level of devastation when it unleashed two on Nagasaki and Hiroshima, Japan. Other nations scrambled to catch up; in the hands of just one country, this technology could arguably give that country control over the rest of the world.

Within eight years, the USSR had its own nuclear weapon – the hydrogen bomb. The ideological conflict between capitalism and communism sustained tensions between the U.S. and the USSR, and this prolonged conflict between the nations became known as the Cold War. From 1947 to 1991, the nations built up their nuclear arms, each expanding its arsenal in pace with the other. It was soon clear that both sides had built and stockpiled enough nuclear warheads that the U.S. and USSR could wipe out each other (and the rest of the world) several times over. They had reached *nuclear parity*, or a state of equally destructive capabilities.

As a result, the nuclear strategy doctrine of Mutual Assured Destruction (MAD) emerged in the mid-1960s. This doctrine was based upon the size of the countries' respective nuclear arsenals and their unwillingness to destroy civilisation. MAD was unique at the time. Never before had two warring nations held the potential to erase humanity with the entry of a few computer codes and the turn of matching keys. Ironically, it was this powerful potential that guaranteed the world's safety: *Nuclear capability was a deterrent against nuclear war.*

Because the U.S. and the USSR both had enough nuclear missiles to clear each other from the map, neither side could strike first. A first strike guaranteed a retaliatory counter-strike from the other side. So launching an attack would be tantamount to suicide – the first striking nation could be certain that its people would be annihilated, too.

The doctrine of MAD guided both sides toward deterrence of nuclear war. It could never be allowed to break out between the two nations. And it virtually guaranteed no conventional war would, either. Eventually, conventional tactics – like non-nuclear missiles, tanks and troops – would run out, and the inevitable conclusion of a nuclear strike would be reached. Since that end was deemed unacceptable by the Soviets and Americans, there was no chance of an engagement that could lead to this conclusion.

But MAD didn't exactly create an atmosphere in which Soviet premiers and American presidents felt like they could shake hands and call the whole thing off. The nations had very little trust in each other – and with good reason. Each side was steadily building its nuclear arsenal to remain an equal party in the MAD doctrine. A **détente**, or uneasy truce, developed between the U.S. and USSR. They were like two gunslinging foes, adrift alone in a life boat, each armed and unwilling to sleep. The situation had to change.

There are two defining characteristics of the doctrine of Mutual Assured Destruction. One, each side must have the nuclear capability to wipe out the other. And two, each side must be convinced the other has the nerve to launch a nuclear strike. In a speech in 1967, Defence Secretary Robert McNamara described how the U.S. achieved nuclear deterrence through MAD:

"We do this by maintaining a highly reliable ability to inflict unacceptable damage upon any single aggressor or combination of aggressors at any time during the course of a strategic nuclear exchange, even after absorbing a surprise first strike." [source: McNamara].

Over time, nuclear delivery became more refined and the nightmare of an all-out nuclear holocaust less realistic. Both the U.S. and USSR invested heavily in technology that directed thermonuclear weapons from mindless, clobbering bombs to precise surgical instruments. Missile guidance systems allowed for more exact strikes, and the placement of missiles around the globe – from allied nations to submarines cruising the world's oceans – created a virtual nuclear minefield. All-out annihilation was replaced by other options for a nuclear strike [source: Battilega].

People began analysing ways that nuclear war could play out. One theory is called the *ladder of escalation*. Under this strategy, one side launches a first strike, followed by a counter-strike from the other side. This exchange continues like a chess game, with each side increasing the level of destruction with each successive strike. For example, targeting civilian populations comes after strikes against military targets [source: Croddy, et al]. **Each strike gives the other the option to back down or return fire.** It's kind of like trading punches with another person; each punch becomes increasingly powerful. The idea is to step up the force little by little until one heavy blow turns out to be the final punch as the weakened opponent backs down. All-out nuclear war, by contrast, is more akin to two parties shooting each other point-blank in the head.

Fans of the 1983 movie *War Games* will recognise this kind of strategy. In the film, a renegade supercomputer, capable of launching an American first strike, mulls over the best way to win a nuclear war. The computer runs through scenarios like a set of infinite games, considering a strike launched from Europe or from nuclear subs, and other endless possibilities. **The computer finds there's no way to win: Each first strike results in a counter-strike and both sides lose.**

Mutual Assured Destruction really does have a basis in games. The same underlying mathematical principles that dictate manoeuvres in games like *Scrabble* and *Monopoly* were used to examine nuclear strategy during the Cold War using game theory. The doctrine of MAD, specifically, shares its basis with a game theory experiment called the *Prisoner's Dilemma*.

A recap, for those unfamiliar: In this scenario, two criminals are apprehended by police and questioned separately. The dilemma comes from each criminal's uncertainty as to what her cohort will do. If one confesses, the other is released but the confessor is punished: if one criminal implicates the other, the snitch will be freed but the other person punished. The *best course of action* in this scenario (or in nuclear war) is inaction. By remaining mute (or unwilling to launch a first strike), neither party can be implicated (or destroyed).

However, when we analyse the *Prisoner's Dilemma* from a mathematical perspective, we obtain that the Nash Equilibrium, supposedly the strategy that humans would play, is (Destroy, Destroy). I suppose poetically you can some up the game theory analysis as follows:

If both sides of the war have nuclear capabilities, and are willing to use them, then the game theory predicts both sides will destroy each other completely. The much preferable option, as the computer Joshua learns in *War Games*, is that the only way to win in nuclear war is not to play.

(Source).

3.3 More International Relations Examples: The Nuclear Capabilities of Iran.

The following article was written in 2012, hence is a bit outdated. The game theoretic analysis, however, is still sound.

Iran's nuclear research program generates global suspicion and concern. We read news about the possibility of an Israeli or a U.S. attack upon Iranian nuclear facilities. The general idea is that if Iran continues its research, and, say, an Israeli attack occurs, Iran can retaliate. Thus, any potential attacker has to foresee the consequences of its action against Iran. Iran then has to assess the magnitude of costs that deter a potential attacker. Yet, the attacker can guess that Iran has strong incentives to misrepresent its incentives in its will and ability to retaliate. Iran can in turn try to guess whether the attacker takes its stand about retaliation seriously.

We can qualify the above interaction as constituting a game, that is, a situation of strategic interdependence. Each decision maker acts in function of actions the other (or others) can take. The best choices of Iran and the attacker depend on their forecasts of each other's behaviour. An Iranian decision shapes the gains an attacker obtains and the costs an attacker suffers, and the attacker's decision in turn shapes the Iranian decision to retaliate or not.

For those of you who are joining us: Game theory studies the behaviour of decision makers in situations of strategic interdependence. Its founders are John Von Neumann and Oskar Morgenstern who published the book *The Theory of Games and Economic Behavior* in 1944. The relevance of the theory for international relations (IR) goes undisputed; it is a truism to assert that states interact by trying to predict other states' reactions to their decisions. Yet one has to apply the theory to IR, because the tool of game theory cannot produce by itself insight about IR.

Game-theory applications to IR take the form of models, that is, the simplification and stylization of states' interactions. Three levels of game theory are of help here. These levels are (1) extensive, (2) strategic, and (3) coalitional forms. In a (1) extensive-form model, the analyst thinks in terms of states presented as players, actions available to players, sequences of players' actions, players' information conditions and preferences, and, finally, outcomes of interactions. In a game at the (2) strategic level, there are nothing but players, players' strategies and preferences over outcomes[14]. The (3) coalitional form is the most abstract level analysis: coalitions or partnerships of players and the values of these coalitions. The majority of IR game models are pitched at the first two levels, as the last level of analysis assumes that cooperation between players is binding. Yet if a state cooperates, it must do so only because of self-interest; not because of a higher authority above states enforcing cooperative agreements. At least, there is no supreme authority over sovereign and co-existing states.

The major advantage of game models comes through disciplined stylizations of international interactions. The *discipline* comes out of precisely defined concepts of players, strategies, actions, preferences, and deductions formally derived from basic assumptions and concepts. Note that all game theorists around the world would agree upon the meaning of central game theory concepts and would derive the same results, for example, conditions for equilibrium existence. For example, if there is a model of Iran-Israel nuclear conflict in extensive form, then the same solution can be found provided that it exists. As a result, game theory becomes a paradigm through the existence of commonly agreed upon concepts and assumptions.

Starting in the 1950s, political scientists found game theory quite useful in their analyses. The 1960s, for example, were prolific years in the field of coalitional bargaining, voting, and coalition formation. Economists discovered how powerful the tool of game theory is much later in 1980s especially through a program called Nash equilibrium refinement[15]. Nevertheless, while game theory became a major staple in economic analyses, there has been no parallel move in the field of international relations. To illustrate, no student who ignores Nash equilibrium can pass a microeconomics course yet no such condition exists for an IR student, say in a course on IR theory. The source

[14]These are what my class have been analysing mathematically so far.

[15]Almost like adding addendums to Nash's theories to take into account real life features, economists twisted Nash's theories slightly to suit their needs. See here for an example of such refinements.

of the difference is the tolerance for and the use of mathematics in economics.

The IR discipline is divided into many islands of theoretical approaches ranging from realism to liberalism, constructivism, and critical IR works. A majority of IR students would think that game theory is of use only if one frames an international interaction in *realist* terms like power, motives for expansion, and maximization of self-interest. Well, this is completely wrong. Preferences of players, the driving force of game models, are assessed through players' ideas, wishes, and desires. To illustrate, game-theoretic models do not require that all states are cast as selfish egoists; on the contrary, states can be presented as altruistic players[16]. Moreover, repeated games contain rigorous reflections of variables such as inter-subjectivity, shared knowledge, practices, and norms which are of interest for constructivists. Indeed, as long as there is room for ideas and beliefs in theories of preference formation, dynamic game models can dwell into areas where social constructions are argued to play a major role.

Naturally, no one has an obligation to learn game theory. Yet, those motivated students can try to master it and enjoy its power in generating explanations. However, **students must realize that game models are abstractions; they are not equivalent to real interactions.** If they construct a game, they must be aware that the assumptions of the model lead to constrained and stylized explanations. There is, in fact, a trade-off: game theory cannot help students to understand and predict international phenomena if it has no connection with empirical facts, and, if too many observed details are included in the model, deductions become intractable.

In gist, the creativity of modelers is of utmost importance in using game theory. The construction of correspondence between abstract concepts and empirical observations is not an easy task. Once the work is finished, and results are obtained, one can compare game-theoretic explanations with explanations other research tools and approaches to IR generate. It is possible that readers find game-theoretic explanations not as enriching as those other theories yield. Nevertheless, the game theorist has an upper hand: she can be certain that the model implies the explanation provided that assumptions are justified and she correctly derives conditions for equilibrium, equilibria or even no equilibrium. In addition, she can also develop her analysis in a deductive and a rigorous manner so that her findings inform users of other approaches.

[16]See Martin J. Osborne, *An Introduction to Game Theory* (New York, Oxford: Oxford University Press, 2004), p. 27.

Take, for example, the problem of Iran's nuclear research activities constituting yet another source of friction between Iran and Israel, and, opt for the simplest possible model at strategic level: a 2×2 game. Assume Israel has two strategies: attack and do not attack. Assume also that Iran has two strategies: stop nuclear research and do not stop. Hence, we have two players and each player has two strategies. The outcome matrix becomes the following:

		Israel	
		Attack	Do Not Attack
Iran	Stop	Outcome 1	Outcome 2
	Do Not Stop	Outcome 3	Outcome 4

To obtain a payoff matrix we need to specify both countries' preferences over these outcomes.

The most convenient way to model the interaction is to specify players' preferences along their primary and secondary objectives.

Assume that Iran's main objective is to become a nuclear power and Israel's main objective is the inverse. Suppose that an Israeli attack cannot destroy all Iranian facilities, Iran mostly prefers outcomes 3 and 4 as compared to outcomes 1 and 2. The decision "stop" prevents Iran to attain its most preferred objective. Thus, for Iran, we have

$$\{\text{Outcome 3, Outcome 4}\} > \{\text{Outcome 1, Outcome 2}\}.$$

Suppose also that Iran prefers outcome 4 to outcome 3 and outcome 2 to outcome 1 as it prefers no Israeli attack; its secondary objective. These assumptions generate the following preference ordering for Iran:

$$\text{Outcome 4} > \text{Outcome 3} > \text{Outcome 2} > \text{Outcome 1}.$$

We now have some ordinal preferences for Iran.

Israel mostly prefers outcomes 1 and 2 as compared to outcomes 3 and 4, as Iran's stop decision leads to the realisation of Israel's main objective: Iran does not become a nuclear power. Thus, for Israel, we have

$$\{\text{Outcome 1, Outcome 2}\} > \{\text{Outcome 3, Outcome 4}\}.$$

Suppose also that Israel prefers outcome 2 to outcome 1 and outcome 4 to outcome 3 as it prefers to avoid a military failure; its secondary objective. These assumptions generate the following preference ordering for Israel:

$$\text{Outcome 2} > \text{Outcome 1} > \text{Outcome 4} > \text{Outcome 3}.$$

We now have some ordinal preferences for Israel.

Now assume also ordinal-level preferences, with 4 indicating the best, 3 the next-best, 2 the next-worst, and 1 the worst outcome for players. The game matrix therefore becomes:

		Israel	
		Attack	Do Not Attack
Iran	Stop	1,3	2,4
	Do Not Stop	3,1	4,2

(The first number in each cell denotes Iran's preference for that outcome and the second number denotes that of Israel.)

We realise that Israel obtains better outcomes by choosing "Do Not Attack" regardless Iran's choices (*Do Not Attack* is the dominant strategy): Israel obtains 4 instead of 3 against Iran's decision "Stop", and, 2 instead of 1 against Iran's decision of "Do Not Stop" by choosing "Do Not Attack". Similarly, Iran obtains better outcomes by choosing "Do Not Stop" regardless Israeli choices: Iran obtains 3 instead of 1 against Israeli decision "Attack" and 4 instead of 2 against Israeli decision of "Do Not Attack" (it is also a dominant strategy). Therefore, the equilibrium is (Do Not Stop, Do Not Attack). Thus, we explain the (at the time) current status quo between Iran and Israel regarding Iranian nuclear research through a drastic simplification: there are only two players, each has two strategies, they simultaneously interact only once, their preferences are ordered according to primary and secondary objectives, and each player strives to obtain highest possible outcome given other's choices.

It can be a challenging exercise for students to change players' primary and secondary objectives yielding a new game. The model asks for additional justifications or amendments; it does not represent the only possible stylization. Nevertheless, it is possible that the equilibrium does not change as a result of new assumptions. This would inform the modeler about the impact of different assumptions upon explanations. Consequently, game theory, as a deductive method, generates the joy and the suspense (may I say the thrill?) of obtaining new explanations for international interactions by changing game rules and assumptions.

(Serdar Guner).

3.4 Can Game Theory Predict When Iran Will Get The Bomb?

First, we will look at a timeline of Iran's nuclear capabilities, courtesy of CNN:

1957 - The United States signs a civil nuclear cooperation agreement with Iran.

1958 - Iran joins the International Atomic Energy Agency[17].

1967 - The Tehran Nuclear Research Center, which includes a small reactor supplied by the United States, opens.

1968 - Iran signs the Nuclear Non-Proliferation Treaty.

Mid-1970s - With United States' backing, Iran begins developing a nuclear power program.

1979 - Iran's Islamic revolution ends Western involvement in the country's nuclear program.

December 1984 - With the aid of China, Iran opens a nuclear research centre in Isfahan.

February 23, 1998 - The United States announces concerns that Iran's nuclear energy program could lead to the development of nuclear weapons.

March 14, 2000 - US President Bill Clinton signs a law that allows sanctions against people and organisations that provide aid to Iran's nuclear program.

February 21, 2003 - IAEA Director General Mohamed ElBaradei visits Iran to survey its nuclear facilities and to encourage Iran to sign a protocol allowing IAEA inspectors greater and faster access to nuclear sites. Iran declines to sign the protocol. ElBaradei says he must accept Iran's statement that its nuclear program is for producing power and not weapons, despite claims of the United States to the contrary.

[17] The International Atomic Energy Agency (IAEA) inspects nuclear and related facilities under safeguard agreements.

June 19, 2003 - The IAEA issues a report saying that Iran appeared to be in compliance with the Non-Proliferation Treaty, but that it needed to be more open about its activities.

August 2003 - The IAEA announces that its inspectors in Iran have found traces of highly enriched uranium at the Natanz uranium enrichment plant. Iran claims the amounts are contamination from equipment bought from other countries. Iran agrees to sign a protocol of the Nuclear Non-Proliferation treaty that allows for unannounced visits to their nuclear facilities and signs it on December 18, 2003.

October 2003 - The Foreign Ministers of Britain, France and Germany visit Tehran, and all parties agree upon measures Iran will take to settle all outstanding issues with the IAEA. Under obligation to the IAEA, Iran releases a dossier on its nuclear activities. However, the report does not contain information on where Iran acquired components for centrifuges used to enrich uranium, a fact the IAEA considers important in determining whether the uranium is to be enriched for weapons.

November 2003 - Iran agrees to halt uranium enrichment as a confidence building measure and accepts IAEA verification of suspension.

December 2003 - Iran signs an Additional Protocol with the IAEA voluntarily agreeing to broader inspections of its nuclear facilities.

February 2004 - A.Q. Khan, "father" of Pakistan's nuclear weapons program, admits to having provided Iran and other countries with uranium-enrichment equipment.

June 1, 2004 - The IAEA states they have found traces of uranium that exceed the amount used for general energy production. Iran admits that it is importing parts for advanced centrifuges that can be used to enrich uranium, but is using the parts to generate electricity.

July 31, 2004 - Iran states that it has resumed production on centrifuge parts used for enriching uranium, but not enrichment activities.

August 8, 2005 - Iran restarts uranium conversion, a step on the way to enrichment, at a nuclear facility, saying it is for peaceful purposes only, and flatly rejects a European offer aimed at ensuring the nation does not seek nuclear weapons.

August 9, 2005 - Iran removes the IAEA seals from its Isfahan nuclear processing facility, opening the uranium conversion plant for full operation. IAEA spokesman Mark Gwozdecky states that the plant "is fully monitored by the IAEA" and "is not a uranium enrichment plant."

September 11, 2005 - Iran's new foreign minister, Manouchehr Mottaki, says the country won't suspend activities at its Isfahan uranium conversion facility and it plans to seek bids for the construction of two more nuclear plants.

January 10, 2006 - Iran resumes research at its Natanz uranium enrichment plant, arguing that doing so is within the terms of an agreement with the IAEA.

January 12, 2006 - Foreign ministers of the EU3 (Great Britain, France, Germany) recommend Iran's referral to the United Nations Security Council over its nuclear program.

January 13, 2006 - Iran's Foreign Minister, Manouchehr Mottaki, states that if Iran is referred, its government under law will be forced to stop some of its cooperation with the IAEA, including random inspections.

February 4, 2006 - President Ahmadinejad orders Iran to end its cooperation with the IAEA.

April 11, 2006 - Hashemi Rafsanjani, Iran's former president, states that Iran has increased the number of functioning centrifuges in its nuclear facilities in Natanz and has produced enriched uranium from them.

August 31, 2006 - The IAEA issues a report on Iran saying the Islamic republic "has not suspended its enrichment activities" despite this day's deadline to do so. Iran can possibly face economic sanctions.

December 23, 2006 - The UN Security Council votes unanimously to impose sanctions against Iran for failing to suspend its nuclear program.

February 22, 2007 - The IAEA issues a statement saying that Iran has not complied with UN Security Council for a freeze of all nuclear activity. Instead, Iran has expanded its uranium enrichment program.

March 24, 2007 - The UN adopts Resolution 1747 which toughens sanctions against Iran. The sanctions include the freezing of assets of 28 individuals and organisations involved in Iran's nuclear and missile programs. About a third of those are linked to the Iranian Revolutionary Guard, an elite military corps.

May 23, 2007 - The IAEA delivers its latest report to the United Nations on Iran's nuclear activities. The report states that not only has Iran failed to end its uranium enrichment program but has in fact expanded activity in that area.

June 21, 2007 - Iran's Interior Minister Mostapha PourMohamedi claims, "Now we have 3,000 centrifuges and have in our warehouses 100 kilograms of enriched uranium. . . . We also have more than 150 tons of raw materials for producing uranium gas."

December 2007 - A US intelligence report finds that Iran abandoned a nuclear weapons program in 2003.

February 20, 2009 - The Institute for Science and International Security (ISIS) reports that Iranian scientists have reached "nuclear weapons breakout capability." The report concludes Iran does not yet have a nuclear weapon but does have enough low-enriched uranium for a single nuclear weapon. An official at the IAEA cautions about drawing such conclusions. The IAEA says Iran's stock of low-enriched uranium would have to be turned into highly enriched uranium (HEU) in order to be weapons-grade material.

February 25, 2009 - Iran runs tests at its Bushehr nuclear power plant using 'dummy' fuel rods, loaded with lead in place of enriched uranium to simulate nuclear fuel. A news release distributed to reporters at the scene states the test measured the "pressure, temperature and flow rate" of the facility to make sure they were at appropriate levels. Officials say the next test will use enriched uranium, but it's not clear when the test will be held or when the facility will be fully operational.

August 12, 2009 - The article on game theory we will be exploring is written.

September 21, 2009 - In a letter to the IAEA, Iran reveals the existence of a second nuclear facility. It is located underground at a military base, near the city of Qom.

October 25, 2009 - IAEA inspectors make their first visit to Iran's newly disclosed nuclear facility near Qom.

February 18, 2010 - In a statement, the IAEA reports that it believes Iran may be working in secret to develop a nuclear warhead for a missile.

August 21, 2010 - Iran begins fuelling its first nuclear energy plant, in the city of Bushehr.

December 5, 2010 - Ali Akbar Salehi, Iran's atomic chief and acting foreign minister, announces that Iran's nuclear program is self-sufficient and that Iran has begun producing *yellowcake*, an intermediate stage in processing uranium.

January 8, 2011 - Ali Akbar Salehi reports that Iran can now create its own nuclear fuel plates and rods.

September 4, 2011 - Iran announces that its Bushehr nuclear power plant joined the electric grid September 3, making it the first Middle Eastern country to produce commercial electricity from atomic reactors.

September 5, 2011 - In response to Iran's nuclear chief stating that Iran will give the IAEA "full supervision" of its nuclear program for five years if UN sanctions are lifted, the European Union says that Iran must first comply with international obligations.

November 8, 2011 - The IAEA releases a report saying that it has "serious concerns" and "credible" information that Iran may be developing nuclear weapons.

January 9, 2012 - The IAEA confirms that uranium enrichment has begun at the Fordo nuclear facility in the Qom province in northern Iran.

January 23, 2012 - The European Union announces it will ban the import of Iranian crude oil and petroleum products.

January 29, 2012 - A six-member delegation from the IAEA arrives in Tehran for a three-day visit, shortly after the European Union imposes new sanctions aimed at cutting off funding to the nuclear program.

January 31, 2012 - In Senate testimony James Clapper, Director of National Intelligence, says there's no evidence Iran is building a nuclear bomb. CIA director David Petraeus agrees.

February 15, 2012 - Iran loads the first domestically-produced nuclear fuel rods into the Tehran research reactor.

February 21, 2012 - After two days of talks in Iran about the country's nuclear program, the IAEA expresses disappointment that no progress was made and that their request to visit the Parchin military base was denied.

March 28, 2012 - Discussions regarding Iran's nuclear future stall.

April 14, 2012 - Talks resume between Iran and six world powers over Iranian nuclear ambitions in Istanbul, Turkey.

May 25, 2012 - An IAEA report finds that environmental samples taken at the Fordo fuel enrichment plant near the city of Qom have enrichment levels of up to 27%, higher than the previous level of 20%.

June 18-19, 2012 - A meeting is held between Iran and the P5+1 (United States, France, Russia, China, Great Britain and Germany) in Moscow. No agreement is reached.

June 28, 2012 - Saeed Jalili writes to European Union foreign policy chief Catherine Ashton warning world powers to avoid "unconstructive measures" such as the oil embargo that's about to go into effect and that was agreed upon by the EU in January.

July 1, 2012 - A full embargo of Iranian oil from the European Union takes effect.

August 30, 2012 - A United Nations report finds that Iran has stepped up its production of high-grade enriched uranium and has re-landscaped Parchin, one of its military bases, in an apparent effort to hamper a UN inquiry into the country's nuclear program.

September 24, 2013 - At a speech at the UN General Assembly Iranian President Hassan Rouhani says "Nuclear weapons and other weapons of mass destruction have no place in Iran's security and defence doctrine, and contradict our fundamental religious and ethical convictions."

October 16, 2013 - The latest discussions between Iran and the six world powers centre on a proposal put forth by Iran to recognise the peaceful nature of its nuclear energy pursuits. The meeting is described as "substantive and forward-looking."

November 24, 2013 - Six world powers and Iran reach an agreement over Iran's nuclear program. The deal calls on Iran to limit its nuclear activities in return for lighter sanctions.

January 12, 2014 - It is announced that Iran will begin eliminating some of its uranium stockpile on January 20.

January 20, 2014 - Iran's nuclear spokesman Behrouz Kamalvandi tells state-run news agency IRNA that Iran has started suspending high levels of uranium enrichment. The European Union announces that it has suspended certain sanctions against Iran for six months.[18]

[18] Why do you think Iran is suddenly happy to relax a bit from striving for nuclear capabilities?

February 20, 2014 - Following talks in Vienna, EU foreign policy chief Catherine Ashton and Iranian Foreign Minister Mohammad Javad Zarif announce that a deal on the framework for comprehensive negotiations over Tehran's nuclear program has been reached.

November 24, 2014 - The deadline for a final nuclear agreement between Iran and the UN Security Council's P5+1 countries (the United States, Russia, China, France, Britain and Germany) has been set for July 1, 2015.

April 2, 2015 - Negotiators from Iran, the United States, China, Germany, France, Britain and Russia reach a framework for an agreement on Iran's nuclear capabilities, which includes reducing its stockpile of low-enriched uranium by 98%. The deadline for the complete agreement is July 1.

April 9, 2015 - Iranian President Hassan Rouhani announces that Iran will only sign a final nuclear agreement if economic sanctions are lifted on the first day of implementation.

July 14, 2015 - A deal is reached on Iran's nuclear program. The deal reduces the number of Iranian centrifuges by two-thirds. It places bans on enrichment at key facilities, and limits uranium research and development to the Natanz facility.

July 20, 2015 - The UN Security Council endorses the nuclear deal.

January 16, 2016 - International Atomic Energy Agency Director General Yukiya Amano says Iran has completed all the necessary steps agreed under the nuclear deal, and that all participants can begin implementing the Joint Comprehensive Plan of Action (JCPOA).

March 8-9, 2016 - Iran test-fires two Qadr ballistic missiles during a large-scale military drill, according to Iran's state-run Press TV. US officials say that the tests do not violate the nuclear agreement (JCPOA), but are very likely in breach of a UN resolution calling on Iran not to undertake ballistic missile activity.

January 29, 2017 - Iran launches a medium-range ballistic missile, its first missile test since Donald Trump became US president, but the test fails, according to information given to CNN by a US defence official. National Security Adviser Michael Flynn says the US has put "Iran on notice."

February 3, 2017 - In reaction to the January 29 missile test, the US Treasury Department says it is applying sanctions on 25 individuals and companies connected to Iran's ballistic missile program and those providing support

> to Iran's Islamic Revolutionary Guard Corps' Qods Force. National Security Adviser Michael Flynn says the tests were in defiance of a UN Security Council resolution that bars Iran from taking steps on a ballistic missile program capable of launching nuclear weapons.

Other sources of information:

- Wikipedia page for *Iran and weapons of mass destruction.*
- Nuclear Threat Initiative factfile on Iran.
- Arms Control Association factfile: *Nuclear Weapons: Who Has What at a Glance.*

Article in the New York Times, 2009.

Since this has been written, there have updates to the situation.

- Iran poised to increase uranium enrichment at higher levels, 2019.
- EU powers resist calls for Iran sanctions after breach of nuclear deal.

Is Iran going to build a bomb?
Many people wonder, but Bruce Bueno de Mesquita claims to have the answer.

Bueno de Mesquita is one of the world's most prominent applied game theorists. A professor at New York University and a senior fellow at the Hoover Institution at Stanford, he is well known academically for his work on 'political survival' – or how leaders build coalitions to stay in power. But among national-security types and corporate decision makers, he is even better known for his prognostications. For 29 years, Bueno de Mesquita has been developing and honing a computer model that predicts the outcome of any situation in which parties can be described as trying to persuade or coerce one another.

Since the early 1980s, C.I.A. officials have hired him to perform more than a thousand predictions; a study by the C.I.A., now declassified, found that Bueno de Mesquita's predictions hit the bull's-eye twice as often as its own analysts did.

Last year (2008), Bueno de Mesquita decided to forecast whether Iran would build a nuclear bomb. With the help of his undergraduate class at N.Y.U., he researched the primary power brokers inside and outside the country - anyone with a stake in Iran's nuclear future. Once he had the information he needed, he fed it into his computer model and had an answer in a few minutes.

In June, I visited Bueno de Mesquita at his San Francisco home to see the results. A tall man with a slab of gray hair, Bueno de Mesquita, who is 62, welcomed me with painstakingly prepared cups of espresso. Then he pulled out his beat-up I.B.M. laptop - so old that the lettering on the A, S, D and E keys was worn off - and showed me a spreadsheet that summarized Iran's future.

The spreadsheet included almost 90 players. Some were people, like the Iranian president, Mahmoud Ahmadinejad, and Supreme Leader Ali Khamenei; others were groups, like the U.N. Security Council and Iran's religious radicals.

Next to each player, a number represented one variable in Bueno de Mesquita's model: the extent to which a player wanted Iran to have the ability to make nuclear weapons. The scale went from 0 to 200, with 0 being 'no nuclear capacity at all' and 200 representing a test of a nuclear missile.

At the beginning of the simulation, the positions were what you would expect. The United States and Israel and most of Europe wanted Iran to have virtually no nuclear capacity, so their *preferred outcomes were close to zero*. In contrast, the Iranian hard-liners were aggressive. "This is not only 'Build a bomb'," Bueno de Mesquita said, characterizing their position. "It's probably: 'We should test a bomb'."

But as the computer model ran forward in time, through 2009 and into 2010, positions shifted. American and Israeli national-security players grudgingly accepted that they could tolerate Iran having some civilian nuclear-energy capacity. Ahmadinejad, Khamenei and the religious radicals wavered; then, as the model reached our present day, their *power* - another variable in Bueno de Mesquita's model - sagged significantly.

Amid the thousands of rows on the spreadsheet, there's one called **Forecast**. It consists of a single number that represents the most likely consensus of all the players. It begins at 160 - bomb-making territory - but by next year (2010) settles at 118, where it doesn't move much. "That's the outcome," Bueno de Mesquita said confidently, tapping the screen. What does 118 mean? It means that Iran won't make a nuclear bomb. By early 2010, according to the forecast, Iran will be at the brink of developing one, but then it will stop and go no further. If this computer model is right, all the dire portents we've seen in recent months - the brutal crackdown on protesters, the dubious confessions, Khamenei's accusations of American subterfuge - are masking a tectonic shift.

The moderates are winning, even if we cannot see that yet. Could this possibly be what will happen? Certainly Bueno de Mesquita has his critics, who argue that the proprietary software he uses can't be trusted and may cast doubt on the larger enterprise of making predictions. But he has published a large number of startlingly precise predictions that turned out to be accurate, many of them in peer-reviewed academic journals. For example, five years before Ayatollah Khomeini died in 1989, Bueno de Mesquita predicted in the journal *PS* that Khomeini would be succeeded by Ali Khamenei (which he was), who himself would be succeeded by a then-less-well-known cleric named Akbar Hashemi Rafsanjani (which he may well be). Last year, he forecast when President Pervez Musharraf of Pakistan would be forced out of office and was accurate to within a month.

In "The Predictioneer's Game", a book coming out next month that was written for a popular audience, Bueno de Mesquita offers dozens more stories of his forecasts. And as for Iran's bomb?

In a year, he said with a wide grin, we'll know if he's right.

(Clive Thompson)

- Bueno de Mesquita's TED talk.
- Iran-US Nuclear Standoff: A Game-Theoretic Approach.

4 Debate: Should the U.S. have invaded Iraq?

We will now split the class into two groups to debate the topic *should the U.S. have invaded Iraq?* This has ties to international relations and involves careful thinking and strategy. The game theory analysis we applied in the last section and the Theory of Moves in the Cold War section will be crucial for your arguments.

First, we will look at a history of the last century of involvement of the U.S. in Iraqi affairs. This article by Peter Hahn, replicated below, serves as an excellent overview.

Under a cloak of early morning darkness on December 18, 2011, some 500 U.S. soldiers at Camp Adder in southern Iraq boarded 110 military vehicles and drove off quietly into the night, without having notified their local Iraqi colleagues of their departure. On heightened alert, the convoy manoeuvred steadily to the south and reached the border of Kuwait some five hours later.

This departure of the 3rd Brigade Combat Team of the 1st Cavalry Division of the U.S. Army - conducted in secrecy in hope of avoiding any opportunistic attacks by local adversaries - marked the end of a nearly nine-year-long U.S. military adventure in Iraq.

Although the final convoy departed Iraq without incident, it left behind a legacy of a war that was controversial in origin, costly to Iraqi civilians and American soldiers, and inconclusive in outcome.

The 2003 U.S. military invasion of Iraq and the extended occupation that followed were certainly the most dramatic and significant events in the long history of U.S. relations with Iraq. During the nine decades since Iraq was established as a separate state in the aftermath of World War I, the policy of the United States towards it can be divided into five phases.

In each period, the United States pursued distinct goals in Iraq - goals that reflected the growing interest of the United States in the Middle East, the increasing political and military influence of Iraq, and the evolution of U.S. interests in a rapidly changing international context.

I. Genesis of U.S. - Iraqi Relations, to 1958.

Prior to World War II, the U.S. government took very little interest in Mesopotamia (Greek for "land between the rivers," in reference to the basin between the Tigris and the Euphrates, and a name used before World War I for the territory that generally formed modern Iraq).

The first Americans to encounter the region were evangelical Christian missionaries who swarmed across it beginning in the 1830s and who built hundreds of churches, schools, and medical facilities by the turn of the twentieth century. In 1880-1920, archaeologists from American universities conducted field work in Mesopotamia in the hope of discovering physical artefacts that would corroborate Biblical history.

U.S. oil corporations began probing Mesopotamia for commercial opportunities in the 1910s, gaining a 23.75 percent share in the Iraq Petroleum Company (IPC) in 1928. Within a decade, the IPC discovered a massive oil field near Kirkuk and built a network of wells, pipelines, and production facilities that earned it considerable wealth.

U.S. government involvement in early Iraq was limited. President Woodrow Wilson envisioned a liberal post-World War I political system that would include self-determination for Iraqis and other peoples of the former Ottoman Empire, but he was unable to promote that vision effectively.

In the 1920s and 1930s, U.S. diplomats generally deferred to British officials, who managed Iraq as a League of Nations mandate, demarcated its national borders, and built it into a pro-Western monarchy.

When a threat developed that Nazi Germany might gain political dominance in Baghdad during World War II, U.S. diplomats endorsed the British military suppression of Rashid Ali al-Gailani, a pro-Nazi Iraqi who briefly occupied the position of prime minister. With American backing, the British restored the monarchy, which cooperated with Allied war aims and strategy.

Post-World War II international dynamics gradually drew the United States into a deeper political relationship with Iraq. The onset of the Cold War raised fears in Washington about Soviet expansionism into the Middle East and generated a determination among American leaders to prevent the spread of communism in Iraq.

Financially drained by the world war, Britain proved unable to maintain its position of imperial dominance in the country. Intra-regional tensions, most notably the conflict over Palestine that erupted as the first Arab-Israeli War of 1948-49, also destabilized the region. The emergence of anti-Western nationalism—a reaction to the legacy of British imperialism and U.S. support for Israel, among other factors—undermined the local popularity of the pro-Western monarchy in Baghdad.

In the late 1940s and 1950s, U.S. officials sought to stabilize Iraq. They helped to negotiate a withdrawal of Iraqi military forces from the Palestinian theater as part of a broader plan to end the first Arab-Israeli war. They encouraged the IPC to increase oil production and to share a larger portion of revenues with the Iraqi government. They provided economic and military aid to the Iraqi government.

By 1955, the United States enlisted Iraq as a charter member of the Baghdad Pact, an anti-Soviet defense partnership linking Iraq, Iran, Pakistan, Turkey, and Britain, with informal U.S. backing.

Briefly, it appeared that the United States had found a formula for ensuring the long-term stability and anti-communism of Iraq.

But that appearance evaporated quickly in July 1958, when a coalition of Iraqi military officers, disillusioned by the monarchy's subservience to the West and inspired by revolutionary leader Gamal Abdel Nasser of Egypt, overthrew the king in a bloody coup d'état and instituted a new regime with a distinctly anti-western flavor.

In reaction, President Eisenhower sent U.S. Marines into Lebanon to avert a copycat rebellion there, but he rejected the notion of military intervention to reverse the revolution in Baghdad as too difficult tactically and too risky politically.

The Iraqi revolution of 1958 clearly marked the failure of the U.S. quest to align the pro-Western, British-built, royalist government of Iraq on the Western axis in the Cold War.

II. Managing Chronic Instability, 1958-1979.

The second phase of U.S.-Iraqi relations was defined by the political instability in Baghdad that came in the wake of the fall of the Iraqi monarchy in 1958.

The revolution of 1958 was followed by others in 1963, 1968, and 1979. Other revolts reportedly were attempted along the way and political and ethnic-cultural conflicts generated persistent strife throughout the era.

Nationalists aiming to remove the vestiges of foreign imperialism clashed with indigenous communists who sought political influence. The Kurdish population of northern Iraq resisted the authority of Arabs in Baghdad.

Although internally unstable, Iraq emerged as an independent power on the international stage. Its government pursued neutralism in the Cold War and flirted with the Soviet Union and other communist states. It also sought political influence among Arab states and contested Egyptian dominance of the Arab community of nations. Iraq remained technically at war and occasionally skirmished with Israel. Management of the delicate Kurdish problem in the 1970s led Baghdad into alternating conflict and cooperation with Iran.

In the 1958-1979 era, the United States pursued interlocking goals in Iraq. On behalf of U.S. political and economic interests in the country and the region, U.S. officials sought a stable political relationship with the government in Baghdad, aimed to prevent the rise of communism within the country and to deny the Soviet Union influence there, and strove to prevent Iraq from becoming a source of regional conflict or war.

U.S. leaders showed little support for democracy in Iraq or the advancement of its people, eschewing any such liberal political goals on behalf of the primary objective of keeping Iraq free of communism.

For several years after the 1958 coup, U.S. officials accrued some successes in achieving its goals. They maintained diplomatic relations, negotiated the peaceful termination of the Baghdad Pact, averted conflict in an Anglo-Iraqi showdown over Kuwait in 1961, dispensed foreign aid to Iraq, and promoted business opportunities there. In light of evidence that the Soviet Union backed Iraqi Kurds, officials in Washington did nothing to alleviate the Iraqi suppression of that ethnic group.

Nonetheless, U.S.-Iraqi relations declined in the late 1960s.

Iraq severed diplomatic relations in 1967 because it considered the United States complicit in Israeli military conquests during the so-called Six Day War of June 1967. In the early 1970s, Iraq nationalized U.S. petroleum interests and partnered with the Soviet Union to develop its oil capacity.

U.S. officials covertly equipped Kurdish rebels in order to weaken the Iraqi government. Although Iraq neutralized the Kurdish problem through diplomacy with Iran, it criticized foreign powers that backed the Kurds and it displayed renewed anti-U.S. tendencies in its approach to Arab-Israeli issues in the late 1970s.

III. The Initial Challenge of Saddam Hussein, 1979-1989.

The third phase in U.S.-Iraqi relations opened in 1979, when Saddam Hussein seized power in Baghdad. Quickly, Hussein brutally suppressed all domestic rivals and thereby built internal stability in Baghdad, ending decades of political turmoil.

A secularist, Hussein also positioned himself as a vital bulwark against Islamic fundamentalism in Iran, where the Ayatollah Ruhollah Khomeini took power in 1979 and declared an intention to export his revolutionary ideals across the region.

Mounting tension between the two gulf powers erupted into war in September 1980, when Hussein ordered the Iraqi army to launch a full-scale invasion of Iran. Iraq initially occupied 10,000 square miles of Iranian territory before Iran stymied the Iraqi thrust. Iran then gradually recaptured its territory, leading to a stalemate in the battle front by 1982.

A series of massive land offensives proved to be ineffective at breaking the deadlock. Yet the war ground on, widened by missile attacks on cities and by mutual assaults on oil tankers on the Gulf. By 1988, the two states together counted more than one million casualties.

President Ronald Reagan gradually led the United States into involvement in the Iran-Iraq War. Initially, Reagan continued the policy he inherited from Jimmy Carter of practicing strict neutrality in the conflict. By 1982, however, the government in Washington began to shift toward a position of supporting Iraq.

Iran's military advances worried U.S. officials that it might gain political influence across the region and its support of anti-American kidnappers in Lebanon soiled its reputation in the West. Despite Hussein's political despotism, U.S. leaders reinterpreted Iraq as a more benign power and as a vital bulwark against Iranian expansionism.

Thus the Reagan Administration provided Iraq with economic aid, restored diplomatic relations, shared intelligence information about Iranian military forces, and otherwise engaged in what it called a "tilt" toward Iraq designed to ensure its survival. U.S. officials also suspended their protests of Iraq's use of weapons of mass destruction against Iranian troops and domestic rivals.

By 1987, the Reagan Administration even assumed limited military involvement in the war on behalf of Iraq. When Iran attacked oil tankers carrying Iraqi oil to world markets, Reagan ordered the U.S. Navy to patrol the Gulf and protect those tankers. Armed clashes occurred between U.S. and Iranian naval vessels, peaking in late 1987 and mid-1988.

Taking advantage of the relaxation of Cold War tensions, Reagan also worked with Soviet and other world leaders to fashion a United Nations ceasefire resolution that provided a legal framework for ending the hostilities. Iraq promptly accepted the ceasefire but Iran refused, demanding that Iraq first must agree to pay war reparations. Pressured by the U.S. Navy, however, Khomeini eventually accepted the ceasefire in July 1988.

From the U.S. perspective, the Iran-Iraq ceasefire promised to restore a semblance of stability to the Gulf region for the first time in a decade. Peace on the battlefields would end the bloodletting between the two belligerents and restore lucrative commerce. At the same time, the dramatic improvement in U.S.-Soviet relations diminished the traditional U.S. concern that communism would sweep across the region.

With Khomeini contained, U.S. officials hoped that Saddam Hussein would lead his country and the Middle East into an era of peace, prosperity, and moderation. Yet, U.S. officials refrained from addressing Hussein's dreadful record of human rights abuses, his aggressive tendencies, and his political despotism; nor did they take steps to curb the Western thirst for Middle East oil.

Subsequent events would demonstrate that such U.S. officials unwisely built a Middle East strategy on the unstable foundation of the Hussein regime.

IV. The Gulf War and Containment, 1989-2003.

The fourth era in U.S. policy toward Iraq featured a short, indecisive war between the two states followed by a "long decade" of consequential complications.

The military clash originated in Saddam Hussein's decision, in the aftermath of the Iran-Iraq War, to seek territorial and economic gains at the expense of Kuwait. In 1989 and 1990, Hussein signaled a growing intention to use force to against the tiny emirate.

Hussein's aggressiveness was prompted by multiple incentives: a desire to capture lucrative oil assets and thus relieve the financial burdens incurred in the war against Iran; a quest to achieve stature among neighboring leaders and to rally domestic public opinion behind his regime; and a hope of capturing land that, many Iraqis believed, had been misappropriated to Kuwait decades before.

The George H.W. Bush administration reacted to the mounting tensions by using the relatively stable relationship that emerged during the 1980s as a brake on Iraqi recklessness. Viewing Iraq as an important counterweight against Iranian expansionism, Bush offered political friendship and economic incentives to lure Hussein into proper behaviour.

When tensions rose and Hussein moved 100,000 troops to the Kuwait border, Bush also bolstered the U.S. naval presence in the Gulf and warned Hussein against instigating military action.

Yet Bush continued to deal with Hussein constructively—while ignoring his abysmal human rights and foreign policy records—on the calculation that firmer measures might actually provoke the very aggressive behavior that the United States hoped to prevent.

Iraq's full-scale military invasion of Kuwait on August 2, 1990 clearly demonstrated Hussein's reckless aggressiveness and the futility of Bush administration efforts to deal with him on friendly terms.

As Iraqi units quickly overran the country, U.S. officials resolved to contest the occupation. If left unchallenged, U.S. officials feared, Hussein might continue his military advance into Saudi Arabia. They further reasoned that allowing Hussein to consolidate his hold on Kuwait would garner him enormous political prestige and economic wealth and destabilize the international order that was emerging in the post-Cold War era.

President Bush resolved that he would take necessary steps, up to and including military force, to reverse the Iraqi conquest of Kuwait. And his decision to contest Iraqi expansionism resulted in two strategic initiatives, one centering on deterrence and the second on military action.

First, under Operation Desert Shield, Bush positioned American soldiers in Saudi Arabia as a deterrent against any Iraqi military move into territory beyond occupied Kuwait. Second, in partnership with numerous allies, Bush amassed military forces along the borders of Iraq and Kuwait as pressure on Hussein to abandon Kuwait.

When Hussein refused to leave, the allied militaries launched Operation Desert Storm in January 1991, featuring about five weeks of punishing aerial assaults on Iraqi military, political, and communications targets followed by a ground invasion that liberated Kuwait from Iraqi control.

Bush then made the important and controversial decision to halt his forward advance after the liberation of Kuwait, resisting the temptation to occupy Iraq and depose Hussein. Bush reasoned that a march to Baghdad would fragment his international alliance, exceed the mandate authorized by the United Nations, incur unacceptable U.S. casualties, and lead to a costly, prolonged occupation.

The U.S. president also called for an insurrection against Hussein from within Iraq's Sunni elite, but this move backfired badly, as Kurds and Shiites rebelled instead, prompting a brutal Sunni repression that actually bolstered Hussein's domestic position and power.

As the postwar situation stabilized, Bush and his Oval Office successor

William J. Clinton gradually imposed a multi-faceted containment policy against Iraq.

Under Operations Northern Watch and Southern Watch, they established "no-fly zones" over Iraqi territory north of the 36th parallel and south of the 31st (eventually 33rd) parallel, designed to protect Iraq's Kurdish and Shiite populations from military repression and to prevent Hussein from massing his army on his international borders.

U.S. leaders also persuaded the United Nations to maintain the international financial restrictions imposed during the Gulf War until Hussein complied with all U.N. resolutions, including one calling for Iraq to eliminate its weapons of mass destruction (WMD).

U.S. officials also promoted international inspections of Iraqi military and scientific facilities designed to ensure compliance with the disarmament expectations. Both U.S. presidents also used occasional military strikes to punish Iraq for violating the U.N. resolutions, challenging Western warplanes, or inhibiting arms inspections. They hoped essentially to keep Hussein's power in check until his capacity and inclination for trouble-making eroded.

The containment policy, which lasted until the U.S. invasion of Iraq in 2003, achieved its immediate goal. Although Hussein remained in power in Baghdad, he proved unable to provoke another regional conflict, attack his own Kurdish or Shiite peoples living under the protection of Western military aircraft, or down a single one of those aircraft. The Iraqi economy remained stressed.

By hindering international weapons inspections, Hussein stoked fear that he again was developing WMD, but in reality - as confirmed by Western arms inspectors after 2003 - Iraq's WMD program remained dysfunctional and impotent.

These achievements notwithstanding, the containment policy had an uncertain long-term prognosis.

As time passed, the no-fly zones became politically problematic, as Hussein exploited the situation to bolster his domestic political authority and to win world sympathy for the civilian victims of Western airstrikes. Effective arms inspections ended in December 1998. Hussein blamed the suffering of his people on the economic sanctions (rather than his own non-compliance with U.N. resolutions), and such powers as France and Russia wavered in their commitment to enforce sanctions.

In 1998, the terrorist Osama bin Laden cited the U.S. assaults on Iraq from airbases in Saudi Arabia as one cause of his declaration of war against the United States. Clinton bolstered containment in 1998 by embracing the concept of “regime change” - meaning that he would favor the overthrow of Hussein - but even that step had limited ability to guarantee security interests.

Whether the enhanced containment policy would have worked remains a matter of speculation. In hindsight, however, one could reasonably conclude that the maintenance of the containment approach into the new century had a fair chance of preserving essential U.S. interests in the Middle East during Hussein’s lifetime at a small fraction of the costs incurred in the alternative approach implemented by Clinton’s successor in the Oval Office.

V. War and Reconstruction, 2003-2011.

The fifth era of U.S. policy toward Iraq centered on war and reconstruction.

President George W. Bush, unnerved by the September 11, 2001 terrorist attacks on the United States, launched a military invasion of Iraq designed to destroy Saddam Hussein’s brutal regime.

Insecurity stemming from the 9/11 assaults, which was compounded by a series of anthrax attacks inside the United States in late 2001, led Bush to reinterpret Saddam Hussein - given his legacy of military expansionism and his apparent efforts to restore his WMD capabilities - as a dire threat to American security.

Hawks such as Vice President Dick Cheney and Secretary of Defense Donald Rumsfeld encouraged this reinterpretation, gaining the president’s ear at the expense of Secretary of State Colin Powell and other advisers who were reluctant to wage war. Shell-shocked by the terrorist attacks of late 2001, Congress and the American people gave the president wide latitude to pursue a policy in Iraq centered on ousting Hussein by any means including force.

For 18 months following the 9/11 attacks, the Bush administration gradually led the United States to the brink of war. Speeches by leading officials portrayed the Hussein regime as a mortal danger to the security of the United States and other countries by suggesting that Iraq would likely supply WMD to terrorist groups, with catastrophic consequences. Administration officials also argued that the containment policy launched in 1991 had faltered, enabling Hussein to restore his antebellum capacity to do harm to his neighbors and his own people.

The United States secured U.N. Security Council resolution 1441, which censured Iraqi behavior and warned of serious consequences if it remained defiant. (The United States later claimed that this resolution provided a legal basis for war, a claim that France and other powers disputed.)

The Bush administration openly doubted the assurances of U.N. officials, who hastily resumed arms inspections in Iraq in an effort to avert war, that Iraq was free of WMD. U.S. leaders also rebuffed the advice of other countries, including such allies as France and Germany, that war was unnecessary and improper.

The build-up to war climaxed in early 2003 when the United States invaded Iraq.

On March 17, the Bush Administration issued an ultimatum to Hussein to leave Iraq within 48 hours or face the wrath of the American military. When Hussein, as expected, defied the ultimatum, Bush ordered the Pentagon to attack Iraq on March 19.

Some 125,000 U.S. soldiers, bolstered by 20,000 British and 500 Australian troops, launched aerial and ground operations that quickly resulted in a military victory. In combat operations lasting some 500 hours, the invading forces defeated and scattered the Iraqi army of some 400,000 soldiers, occupied the country, and demolished its regime, at a cost of 139 U.S. and 33 British fatalities.

The luster of the military victory over Hussein's forces would soon be tarnished by the Bush Administration's flawed policy for the postwar period.

For starters, the post-invasion discovery that Hussein had actually lacked WMD capability eroded U.S. credibility given the administration's emphasis on the WMD threat in the build-up to war.

News about the grotesque abuses of Iraqi detainees by U.S. soldiers at the Abu Ghraib prison further undermined the public image of the United States around the world. Domestic and foreign opponents of the original decision to invade Iraq rallied in criticism of U.S. policies.

The Bush administration also blundered in political decisions about the post-combat phase of the invasion.

In the rush to war, top Pentagon officials generally neglected initiatives in the State Department to plan for postwar occupation. Secretary of Defense Rumsfeld refused to increase the size of the U.S. occupation force, despite requests for more troops from top uniformed officers, and the occupation forces proved unable to stymie a wave of lawlessness and violence that destabilized the country in the weeks following the downfall of Hussein.

The Pentagon sent retired General Jay Garner to Baghdad to organize popular elections for a new government within 90 days, a mission that failed miserably.

In May 2003, President Bush belatedly established the Coalition Provisional Authority (CPA) under former Ambassador L. Paul Bremer in hope of stabilizing the country. But Bremer erred massively when he issued CPA Orders Number 1 and 2, which disbanded the Baath Party and dissolved the entire Iraqi government.

In that dual stroke, Bremer eliminated the viable prospect of using vestiges of the Iraqi administrative infrastructure to govern the country and lead it into a brighter future. Instead, the orders alienated the elite, including many who had shown no loyalty to Hussein, rendering them unemployed and without purpose and thus vulnerable to an emerging anti-U.S. armed insurgency.

Indeed, within months of the military victory over Hussein, the United States faced a debilitating insurgency across Iraq. The armed opposition seemed to have three distinct sources: Sunnis who had been tied to the Hussein regime; Shiite militias, like the Mahdi Army led by Muqtada al-Sadr, who sought to attain political influence in the post-Hussein era; and non-Iraqi Islamists who infiltrated Iraq in pursuit of the opportunity to bloody the American military.

By December 2003, suicide attacks, sniper fire, car bombs, and roadside bombs had killed some 300 U.S. soldiers, more than double the number who died in the initial invasion. The death toll among G.I.s soared past 1,000 by September 2004 and 3,000 by January 2007.

The annual costs to the U.S. Treasury also rose dramatically, from $51 billion in 2003 to $102 billion in 2006. The security situation worsened through 2006, when anti-U.S. attacks occurred at nearly double the frequency and lethality as in 2005.

As the insurgency mounted, the Bush Administration labored to build a democratic government in Iraq and made steady if halting progress.

The first breakthrough came in 2004, when the Bush administration abandoned its initial quest to build a new state on the foundation of Iraqi expatriates, notably Ahmad Chalabi, who had proved woefully inadequate to the challenge. In addition, recognizing that the dominance of the CPA generated political backlash, Bremer dissolved the authority in June 2004 and established a multi-ethnic Iraqi Transitional Government to preside over the establishment of a permanent government.

In January 2005, millions of Iraqis participated in a democratic election that established a 275-member Transitional National Assembly, which set out to write a permanent constitution. A second democratic election, held in December 2005 under the new constitution, established the permanent Council of Representatives (that replaced the Transitional National Assembly) and a coalition government.

The inherent clash between the growing insurgency and the quest to democratize Iraq came to a head in 2006. Domestic critics of the Bush administration - including a growing number of members of his own Republican Party - pressured the president to withdraw immediately from Iraq even if that step would result in complete collapse of the new government. Democrats captured majorities in both the House and Senate in the midterm elections of 2006 and in 2007 the new congressional leaders called for prompt demilitarization of the U.S. effort in Iraq.

President Bush resolved instead to escalate and reform the military mission in Iraq. In a strategic initiative known as the "surge," he increased the number of G.I.s in Iraq from 120,000 to 160,000 and he ordered them to reform their modes of operation from using overwhelming firepower (which caused collateral damage and negative political repercussions) to restraining firepower and engaging in political initiatives designed to gain goodwill.

U.S. forces also skillfully used diplomacy, persuasion, and financial aid to mobilize various Iraqi factions to fight against insurgent groups. By 2008, the surge seemed to succeed. The insurgency faltered and the military and political situations stabilized.

Taking office in January 2009, President Barack Obama gradually terminated the U.S. military presence in Iraq. He ended U.S. combat operations in Iraq in August 2010 and, consistent with a Status of Forces Agreement (SOFA) signed in 2008, he withdrew all combat forces from the country in December 2011.

Obama pledged to transfer responsibility for Iraq's future to the Iraqi people and to engage in regional diplomacy to ease external pressures on the country. By the end of the U.S. occupation, the war in Iraq had left nearly 4,500 U.S. soldiers dead and more than 30,000 wounded and had drained more than $1 trillion from the U.S. Treasury.

As U.S. forces departed the country in late 2011, Iraq's future remained precarious.

Sectarian violence spiked, killing at least 250 civilians within a month of the

U.S. withdrawal.

The democratic foundations of the government teetered, as Prime Minister Nouri al-Maliki, a Shiite, took steps to solidify his influence over Sunni legislators and as Sunni Vice President Tariq al-Hashemi fled to Kurdistan to escape arrest for allegedly having ordered assassinations some years before.

Having endured 24 years of brutal rule by the Hussein regime, Iraq also bore scars of the U.S. invasion, including some 100,000 Iraqis killed and two million displaced in 2003-2011 alone and its financial and physical infrastructures badly stressed.

(A list of further reading materials, with some game theoretic content)

These are the rules we will be following for the debate. This is modelled after the Oxford style of debating.

Motion: The U.S. was right to invade Iraq.

Each side (the Proposition and the Opposition) will nominate 2 speakers - the First (head) speaker and the Second speaker. The debate format is as follows:

1. First speaker for the Proposition (6 minutes).
2. First speaker for the Opposition (6 minutes).
3. Debate from the floor, alternating Proposition/Opposition/Judges (15 minutes).
4. Second speaker for the Opposition (4 minutes).
5. Second speaker for the Proposition (4 minutes).
6. 5 minute pause (for gathering rebuttal arguments).
7. Proposition rebuttal, delivered by First speaker (3 minutes).
8. Opposition rebuttal, delivered by First speaker (3 minutes).
9. Judges deliberation.

Interruptions:

The audience may only interrupt your speech using a Point of Information or a Point of Order.

- Point of Information: The speaker can choose to accept or refuse a point of information. This type of interruption should be used to clarify or question a point of information raised by the speaker, and not to express an opinion.
- Point of Order: Speakers must give way to a point of order. Such an interruption must only be used to draw attention to an abuse of the Forms of the House, such as a slanderous remark that they wish to be withdrawn.

The Judges will be keeping track of the debate using points - awarding points for well thought out, mathematical arguments, and removing points for slanderous remarks, incorrect points and poor form.

5 The Market for Lemons

Article from the economist, 2001.

TODAY'S global economy is a colourful landscape. Brands are ubiquitous, sprouting from billboards, television and magazine advertisements that relentlessly tout the virtues of products. Humans themselves are not immune: American business gurus advise aspiring executives to style themselves as a product, 'Brand You'. Nowadays, everything seems to be a sales pitch.

But brands do help to make the world easier to navigate. A Coke or a Big Mac, say, is almost the same everywhere in the world. The customer knows the quality of a product by its brand. To understand why brands are so valuable an economic innovation - despite being pilloried by the anti-globalisation lobby - you need only imagine what happens when sellers offer a product whose quality a buyer cannot easily judge.

Take the frustratingly familiar problem of buying a used car. Assume that used cars come in two types: those that are in good repair, and duds (or "lemons" as Americans and most economists call them). Suppose further that used-car shoppers would be prepared to pay $20,000 for a good one and $10,000 for a lemon. As for the sellers, lemon-owners require $8,000 to part with their old banger, while the one-owner, careful-driver old lady with the well-maintained estate won't part with hers for less than $17,000. If buyers had the information to tell wheat from chaff, they could strike fair trades with the sellers, the old lady getting a high price and the lemon-owner rather less.

If buyers cannot spot the quality difference, though, as is often the case in the real world, there will be only one market for all used cars, and buyers will be ready to pay only the average price of a good car and a lemon, or $15,000. This is below the $17,000 that good-car owners require; so they will exit the market, leaving only bad cars. This result, when bad quality pushes good quality from the market because of an information gap, is known as "adverse selection". This was the simple but powerful insight of one of this year's laureates, George Akerlof, now a professor at Berkeley, in a seminal 1970 paper.

Theories that deal with similar instances of so-called "asymmetric information" - when one party to a deal knows more than the other - link all three of this year's winners. It seems that a great many markets, including those for shares, labour, insurance and banking, often resemble a used-car sale more closely than a McDonald's restaurant.

Sending signals, setting screens

Information gaps can be costly, as whoever has the least information can never be confident about what is being traded. But there are ways to reduce the cost. In used-car sales, for example, sellers with good cars can show convincingly that theirs is not a dud. Or used-car buyers can devise better strategies for spotting the lemons. This year's other two laureates, Michael Spence of Stanford University and Joseph Stiglitz of Columbia, won their prize for analysing how firms and consumers separate the gems from the lemons in a variety of industries.

Mr Spence's early work focused on how individuals use signalling to communicate their abilities in the labour market. Job applicants, for example, want to distinguish themselves from the mass of other hopefuls. They may try to do this in a number of ways, from a fancy suit to a fancy education. But for signals to be believable, Mr Spence observed, they need to differ substantially in their cost of acquisition. For example, for education to work as a credible signal, it must be harder for less able employees to get. Indeed, even if such an education gives a student no tangible skills - reading classics at Oxford, say - it can still be a useful signal of relative quality to employers.

Signalling is used in many markets, wherever a person, company or government wants to provide information about its intentions or strengths indirectly. Taking on debt might signal that a company is confident about future profits. Brands send valuable signals to consumers precisely because they are costly to create, and thus will not be lightly abused by their creators. Advertising may convey no information other than that the firm can afford to advertise, but that may be all a consumer needs to know to have confidence in it. Perhaps advertising, as a signal, is not money entirely wasted, as some economists argue.

The theory of signalling can also help to explain why companies pay dividends, even though they are less tax-efficient than share-price rises in compensating investors. Dividends, under the signalling theory, serve as a way of highlighting a strong profit outlook.

Mr Stiglitz is the best-known of the three winners, thanks to his outspoken stint as chief economist at the World Bank during the late 1990s. His insights then got plenty of attention, but are not the reason for his Nobel award. Rather, he has been rewarded for theories drawn from the perspective of a used-car buyer - how to get reluctant people to reveal secrets about themselves or their products. For example, how likely are individuals to repay a loan, or to drive their insured car recklessly? He has explained how insurance companies structure their prices to detect the truth about customers, using different deductibles and premiums to classify customers by their level of risk. This sort of screening of customers is also common in the banking industry.

The winners must split the $1m prize three ways, but they are now unlikely ever again to need to use their expertise at buying lemons.

6 Behavioural Economics & The Game Theory of *Star Wars*

Behavioural economics is probably the fastest growing economic field right now. It combines notions from psychology, experimental economics and game theory to create a whole new field.

First, let's cover subgame perfect equilibria, and once more take a look at backward induction.

- A *subgame perfect equilibrium* (or subgame perfect Nash equilibrium) is a refinement of a Nash equilibrium used in dynamic games. **A strategy profile is a subgame perfect equilibrium if it represents a Nash equilibrium of every subgame of the original game.**

- Informally, this means that if the players played any smaller game that consisted of only one part of the larger game, their behaviour would represent a Nash equilibrium of that smaller game.

- More formally, a *subgame perfect equilibrium* is an action profile a^* with the property that **in no subgame** is it the case that

  $$u_i(a_i, a^*_{-i}) > u_i(a^*) \text{ for } a_i \neq a^*_i,$$

  for all players i and actions $a_i \in A_i$.

- A common method for determining subgame perfect equilibria in the case of a finite game is *backward induction*.

- As we did for the Cuban Missile Crisis, one first considers the last actions of the game and determines which actions the final mover should take in each possible circumstance to maximize his/her utility. One then supposes that the last actor will do these actions, and considers the second to last actions, again choosing those that maximize that actor's utility. This process continues until one reaches the first move of the game.

- *The strategies which remain are the set of all subgame perfect equilibria* (for finite-horizon extensive games of perfect information). However, backward induction cannot be applied to games of imperfect or incomplete information because this entails cutting through non-singleton information sets.

- This Wikipedia page has an example of a subgame perfect equilibrium worked out.

Some more information:

- Reinhard Selten proved that any game which can be broken into 'sub-games' containing a subset of all the available choices in the main game will have a subgame perfect Nash Equilibrium strategy (possibly as a mixed strategy giving non-deterministic sub-game decisions)[19].
- One game in which the backward induction solution to determine the subgame perfect equilibira is well known is tic-tac-toe, but in theory even *Go* has such an optimum strategy for all players.

Recall the ultimatum game played on day 1. Let's work out the subgame perfect equilibrium.

There are two versions of the ultimatum game we can consider; the *discrete* and the *continuous* ultimatum game.

1. The discrete game. See the blackboard.
2. The continuous game. See here, page 13 onwards. Spaniel also produces a video explaining this somewhat paradoxical result.

Now let's watch Spaniel's backward induction video. Let's apply these ideas to two examples:

- *Star Wars:* Sith vs. Jedi in *Revenge of the Sith* (Minute 105-120).
- A real-life or fictional example you come up with yourselves.

In the same vein as *Revenge of the Sith*, there are some "baragining blunders" in *The Empire Strikes Back*. And then there's this.

[19] Analogous to Nash's Theorem.

6.1 The Allais Paradox

Suppose somebody offered you a choice between two different vacations. Vacation number one gives you a 50 percent chance of winning a three-week tour of England, France and Italy. Vacation number two offers you a one-week tour of England for sure.

Not surprisingly, the vast majority of people (typically over 80 percent) prefer the one-week tour of England. We almost always choose certainty over risk, and are willing to trade two weeks of vacation for the guarantee of a one-week vacation. A sure thing just seems better than a gamble that might leave us with nothing. But how about this wager:

Vacation number one offers you a 5 percent chance of winning a three week tour of England, France and Italy. Vacation number two gives you a 10 percent chance of winning a one week tour of England.

In this case, most people choose the three-week trip. We figure both vacations are unlikely to happen, so we might as well go for broke on the grand European tour. (People act the same way with lotteries: we typically buy the ticket for the biggest possible prize, regardless of the odds.)

Allais presciently realized that this very popular set of decisions - almost everybody made them - violated the rational assumptions of economics. Instead of making decisions that could be predicted by a few mathematical equations, people acted with frustrating inconsistency. After all, both questions involve 50 percent reductions in probability (from 100 percent to 50 percent, and from 10 percent to 5 percent), and yet generated completely opposite responses. Our choices seemed incoherent.

The Allais paradox was mostly ignored for the next two decades. But then, in the early 1970s, two Israeli psychologists, Daniel Kahneman and Amos Tversky, read about the paradox and were instantly intrigued: they wanted to know why people didn't respond to probabilities in a linear manner. Based upon their conversations with each other, it seemed obvious that people perceived a smaller difference between probabilities of 1 percent and 2 percent than between 0 percent and 1 percent, or between 99 percent and 100 percent. In other words, all changes in risk are not created equal. As Allais had observed decades before, we value complete certainty an inordinate amount.

But why was certainty so attractive? Kahneman and Tversky wanted to understand the psychology behind the paradox. Their breakthrough came by accident. Kahneman had been reading a textbook on economic utility functions, and was puzzled by the way economists explained a particular aspect of our behaviour. When evaluating a gamble - like betting on a hand of poker, or investing in a specific stock - economists assumed that we made the decision by taking into account our wealth as a whole. (Being rational requires factoring in all the relevant information.) But Kahneman realized that this isn't how we think. Gamblers in Las Vegas don't sit around the card table contemplating their complete financial portfolio. Instead, they make quick decisions that depend entirely upon the immediate terms of the gamble. If there is a $100 wager, and you're trying to decide whether or not to ante in with a pair of aces, you probably aren't thinking about the recent performance of your mutual fund, or the value of your home.

But if we don't make decisions based upon a complete set of information, then what are our decisions based upon? Which factors were actually affecting our choices? Kahneman and Tversky realized that people thought about alternative outcomes in terms of gains or losses, and not in terms of states of wealth. The gambler playing poker is only concerned with the chips right in front of him, and the possibility of winning (or losing) that specific amount of money. (The brain is a bounded machine, and can't think about everything at once.) This simple insight led Kahneman and Tversky to start revising the format of their experiments. At the time, they regarded this as nothing but a technical adjustment, a way of making their questionnaires more psychologically realistic.

This minor change in notation soon revealed one of the most important discoveries of their careers. When Kahneman and Tversky framed questions in terms of gains and losses, they immediately realized that people hated losses. In fact, our dislike of losses was largely responsible for our dislike of risk in general. Because we felt the disadvantages of risky decisions (losses) more acutely than the advantages (gains), most risks struck us as bad ideas. This also made options that could be forecast with certainty seem especially alluring, since they were risk-free. As Kahneman and Tversky put it, "In human decision making, losses loom larger than gains." They called this phenomenon 'loss aversion'.

This simple idea has profound implications. For one thing, it reveals a deep bias built into our brain. From the perspective of economics, there is no good reason to weight gains and losses so differently. Opportunity costs (foregone gains) should be treated just like 'out-of-pocket costs' (losses). But they aren't - losses carry a particular emotional sting. Take this imaginary scenario[20]:

[20]One can also revisit the *Blonde-Brunette Problem* from *A Beautiful Mind*.

The U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimates of the consequences of the programs are as follows: If program A is adopted, 200 people will be saved. If program B is adopted, there is a one-third probability that 600 people will be saved and a two-thirds probability that no people will be saved. Which of the two programs would you favor?

When this question was asked to a large sample of physicians, 72 percent chose option A, the safe-and-sure strategy, and only 28 percent chose program B, the risky strategy. In other words, physicians prefer a sure good thing over a gamble that risks utter failure. They are acting just like the people who choose the certain one week tour of England. But what about this scenario:

The U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimates of the consequences of the programs are as follows: If program C is adopted, 400 people will die. If program D is adopted, there is a one-third probability that nobody will die and a two-thirds probability that 600 people will die. Which of the two programs would you favor?

These two different questions examine identical dilemmas. Saving one third of the population is the same as losing two thirds. But when Kahneman and Tversky framed the scenario in terms of losses, physicians reversed their previous decision. Only 22 percent voted for option C, while 78 percent of them opted for option D, the risky strategy that might save everyone. Of course, this is a ridiculous shift in preference, as nothing substantive has changed in the scenario.* But our choices are guided by our feelings, and losses just make us feel bad. Because the coldhearted equations of classical economics neglect emotion, their description of our decisions remained woefully incomplete.

And this returns us to Maurice Allais. It would be easy to dismiss his paradox as a trifling issue, an irrelevant foible of human decision-making. But it actually helped lead to a radical revision of human nature. (Daniel Kahneman went on to win the Nobel Prize in 2002.) We've come to realise that we're not nearly as rational as we like to believe, that the brain is driven by all sorts of inarticulate feelings and pre-programmed instincts. It's worth noting, however, that the modern investigation into our irrationality didn't begin with a brain scan, or with discussions of the amygdala. Instead, it began with a few inconsistent people, making economic decisions about their vacation. Leonard Cohen said it best: "There's a crack in everything - that's how the light gets in." Allais found an important crack.

*Patients exhibit a similar bias: When asked whether they would choose surgery in a hypothetical medical emergency, twice as many people opted to go under the knife when the chance of survival was given as 80 percent than when the chance of death was given as 20 percent.

(Source)

6.2 Tragedy of the Commons

- See Hardin's article.
- This Wikipedia page discusses in in brief. The Examples page is particularly relevant, and has ties to our next section (Evolutionary Biology).
- For something more lighthearted, there is Rose's Comedy of the Commons.

7 Game Theory in Action

7.1 Evolutionary Biology & Game Theory

Two lectures on evolutionary biology delivered by Professor Ben Polak at Yale University.

- Evolutionary Stability: Cooperation, Mutation, and Equilibrium.
- Evolutionary Stability: Social Convention, Aggression, and Cycles.
- Handout on Evolutionary Stability.

7.2 *Magic: The Gathering* & Game Theory

"Magic: The Gathering" is officially the world's most complex game: A new proof with important implications for game theory shows that no algorithm can possibly determine the winner.

Magic: The Gathering is a card game in which wizards cast spells, summon creatures, and exploit magic objects to defeat their opponents.

In the game, two or more players each assemble a deck of 60 cards with varying powers. They choose these decks from a pool of some 20,000 cards created as the game evolved. Though similar to role-playing fantasy games such as Dungeons and Dragons, it has significantly more cards and more complex rules than other card games.

And that raises an interesting question: among real-world games (those that people actually play, as opposed to the hypothetical ones game theorists usually consider), where does Magic fall in complexity?

Today we get an answer thanks to the work of Alex Churchill, an independent researcher and board game designer in Cambridge, UK; Stella Biderman at the Georgia Institute of Technology; and Austin Herrick at the University of Pennsylvania.

His team has measured the computational complexity of the game for the first time by encoding it in a way that can be played by a computer or Turing machine. "This construction establishes that Magic: The Gathering is the most computationally complex real-world game known in the literature," they say.

First, some background. An important task in computer science is to determine whether a problem can be solved in principle. For example, deciding whether two numbers are relatively prime (in other words , whether their largest common divisor is greater than 1) is a task that can be done in a finite number of well-defined steps and so is computable.

In an ordinary game of chess, deciding whether white has a winning strategy is also computable. The process involves testing every possible sequence of moves to see whether white can force a win.

But while both these problems are computable, the resources required to solve them are vastly different.

This is where the notion of computational complexity comes in. This is a ranking based on the resources required to solve the problems.

In this case, deciding whether two numbers are relatively prime can be solved in a number of steps that is proportional to a polynomial function of the input numbers. If the input is x, the most important term in a polynomial function is of the form Cx^n, where C and n are constants. This falls into a class known as P, where P stands for polynomial time.

By contrast, the chess problem must be solved by brute force, and the number of steps this takes increases in proportion to an exponential function of the input. If the input is x, the most important term in an exponential function is of the form Cn^x, where C and n are constants. And as x increases, this becomes bigger much faster than Cx^n. So this falls into a category of greater complexity called EXP, or exponential time.

Beyond this, there are various other categories of varying complexity, and even problems for which there are no algorithms to solve them. These are called non-computable.

Working out which complexity class games fall into is a tricky business. Most real-world games have finite limits on their complexity, such as the size of a game board. And this makes many of them trivial from a complexity point of view. "Most research in algorithmic game theory of real-world games has primarily looked at generalisations of commonly played games rather than the real-world versions of the games," say Churchill and co.

So only a few real-world games are known to have non-trivial complexity. These include Dots-and-Boxes, Jenga, and Tetris. "We believe that no real-world game is known to be harder than NP previous to this work," says Churchill and co.

The new work shows that Magic: the Gathering is significantly more complex. The method is straightforward in principle. Churchill and co begin by translating the powers and properties of each card into a set of steps that can be encoded.

They then play out a game between two players in which the play unfolds in a Turing machine. And finally they show that determining whether one player has a winning strategy is equivalent to the famous halting problem in computer science.

This is the problem of deciding whether a computer program with a specific input will finish running or continue forever. In 1936, Alan Turing proved that no algorithm can determine the answer. In other words, the problem is non-computable.

So Churchill and co's key result is that determining the outcome of a game of Magic is non-computable. "This is the first result showing that there exists a real-world game for which determining the winning strategy is non-computable," they say.

That's interesting work that raises important foundational questions for game theory. For example, Churchill and co say the leading formal theory of games assumes that any game must be computable. "Magic: The Gathering does not fit assumptions commonly made by computer scientists while modelling games," they say.

That suggests computer scientists need to rethink their ideas about games, particularly if they hope to produce a unified computational theory of games. Clearly, Magic represents a fly in the enchanted ointment as far as this is concerned.

(Source)

- Also see the arXiv paper *Magic: The Gathering* is Turing Complete.
- Feature: Introduction to Game Theory in Magic: The Gathering.