The University of Oxford
MSc (Mathematics and Foundations of Computer Science)

FLATNESS IN ALGEBRAIC GEOMETRY

BRIAN TYRRELL

ABSTRACT. In this essay it is our goal to reconcile two definitions of *flat morphism* - one algebraic and one geometric - then demonstrate through examples and non-examples that 'flat' morphisms are aptly named; the dimensions of the fibers of a flat morphism must be constant. We conclude via Grothendieck's Generic Flatness Theorem that flat families are a common presence in algebraic geometry and are well motivated from a topological standpoint.

CONTENTS

Date: Michaelmas 2017.

1. Introduction

Mumford [9] begins his section on flat and smooth morphisms with the remark:

> *The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers.*

In this essay, it is our aim to explore this riddle, untangle the algebra, then recount the favourable (and prayer-answering) properties of flat morphisms. We begin by presenting two definitions of *flat morphism* that we shall later prove are special cases of a more general definition in scheme-theoretic terms.

Definition A. A morphism of projective varieties $f : X \to Y$ is *flat* if the fibers $X_a = f^{-1}(a)$ have the same Hilbert polynomial.

Definition B. A morphism of projective varieties $f : X \to Y$ is *flat* if the induced map on stalks is a flat map of rings.

In §2 we begin by arranging our algebraic backdrop and formalising *Definition B*, then setting out some of the algebraic properties of flat maps. We progress to §3 where we give two examples and two (and a half) non-examples of flat morphisms, and bring in formally for the first time a connection to dimension. The equivalence of *Definition A* and *Definition B* is proven in §4 (for the simplest case) then we retrospectively examine why some of our examples worked while the non-examples failed. The author has titled the theorem giving the equivalence of *Definitions A* and *B* the *Hilbert Polynomial (HP)-Flatness Theorem* (*Theorem 4.1*). Finally in §5 we give the last piece of motivation for flat families in algebraic geometry then wrap up with a remark on the *Generic Flatness Theorem*.

For the reader unaccustomed to notions from homological algebra, the author recommends the classic text *An Introduction to Homological Algebra* by Weibel [13] as a starting point. We will be continuously making reference and leaving algebraic details to Lang [6] throughout the course of this essay, and much of our development follows the treatment of Eisenbud & Harris [4].

2. A Riddle That Comes Out Of Algebra

We shall first lay an algebraic foundation that later shall lead to a geometric presentation of *flatness*.

Definition 2.1. [9, III.10] Let R be a commutative, unital ring. Given an R-module M and an exact sequence of R-modules,

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

M is called *exact* if the sequence

$$0 \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

remains exact.

Note that we are really only asking the functor $- \otimes_R M$ to be left exact, as it is naturally right exact[1]. In particular to this definition we obtain the following lemma:

[1] $N_1 \xrightarrow{f} \twoheadrightarrow N_2$ implies $N_1 \otimes_R M \twoheadrightarrow N_2 \otimes_R M$ via $n_1 \otimes_R m \mapsto f(n_1) \otimes_R m$.

Lemma 2.2. *Let $I \subset R$ be an ideal. The multiplication map $I \otimes_R M \to IM$ is injective if and only if M is a flat R-module.*

Proof. Using *Definition 2.1*, if $N_1 = I$, and $N_2 = R$, then M flat implies

$$0 \to I \otimes_R M \to R \otimes_R M \to N_3 \otimes_R M \to 0$$

is exact, thus $I \otimes_R M \to R \otimes_R M$ is injective. As $I \hookrightarrow R$ was the inclusion map,

$$I \otimes_R M \to R \otimes_R M \cong M \quad \text{via} \quad i \otimes_R m \mapsto i \otimes_R m \cong im,$$

hence $I \otimes_R M \to IM$ is injective.

Conversely, an injectivity assumption of $I \otimes_R M \to IM$ allows us to show the functor $- \otimes_R M$ preserves injective maps $N_1 \hookrightarrow N_2$, making M flat. This is proven using homological algebra (exploiting the fact $\mathrm{Tor}_i(M, N) = 0$ for all $i > 0$ when M is a flat R-module and N is any R-module) however the proof of this would take us too far afield [7, Theorem 1.4]. ■

We conclude that the 'injectivity of tensor multiplication' property is equivalent to flatness, and moreover the derived functor Tor_i allows us to conclude another crucial characterisation of flat modules:

Lemma 2.3. *Let R be a principal ideal domain. Then an R-module M is flat if and only if it is torsion free.*

Proof. As [7] notes, M is torsion free if and only if for all elements $m \in M$, multiplication by m is an injection on M. Hence as R is a PID, if we can show $\mathrm{Tor}_1(M, R/I) = 0$ by the proof of [7, Theorem 1.4] we can conclude $I \otimes_R M \to IM$ is injective thus by *Lemma 2.2*, M is flat.

The calculation of $\mathrm{Tor}_1(M, R/I)$ is quite straightforward: take a projective resolution

$$0 \longrightarrow R \xrightarrow{\pi} R/I \xrightarrow{0} 0$$

of R/I. Apply the (right exact) functor $- \otimes_R M$:

$$R \otimes_R M \xrightarrow{\pi^*} R/I \otimes_R M \xrightarrow{0} 0$$

which is just

$$M \xrightarrow{\pi^*} R/I \otimes_R M \xrightarrow{0} 0$$

then calculate $\mathrm{Tor}_1(M, R/I) = \mathrm{Ker}\,\pi^*/\mathrm{Im}\ 0 = \mathrm{Ker}\,\pi^* = 0$ as π^* embeds M into $R/I \otimes_R M$. We conclude $\mathrm{Tor}_1(M, R/I) = 0$, as required. ■

We shall make use of the following purely algebraic reformulation of flatness in §4:

Corollary 2.4. [2, I §4, Prop. 3] *If R is a domain, and M is a flat R-module, then M is torsion-free.* ■

Let us now define a flat *map*:

Definition 2.5. Let $\phi : R \to S$ be a map of rings. The map ϕ is called *flat* if S is flat as an R-module.

From this the natural definition to make for projective varieties is as follows:

Definition 2.6. Let X, Y be projective varieties. A morphism $f : X \to Y$ is called *flat* if for all points $p \in X$, the induced map on stalks

$$f^{\#} : k[\widehat{Y}]_{f(p)} \to f_* k[\widehat{X}]_p, \quad g \mapsto g \circ f,$$

is flat, i.e. $k[\widehat{X}]_p$ is a flat $k[\widehat{Y}]_{f(p)}$-module.

At this point it is more useful to talk about *schemes* and view them as a generalisation of *varieties* for what we are about to do. For an introduction to scheme theory, there are several well known and classical texts though the author shall be employing definitions and ideas mostly from Eisenbud & Harris [4] and Hartshorne [5].

In order to define flatness in this context we must first make clear what a morphism of schemes is, and how the stalk of a sheaf is formulated.

Definition 2.7. [4, I.2.3] Let $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ be schemes. A *morphism* $\Psi : \mathcal{X} \to \mathcal{Y}$ consists of a pair of maps $(\psi, \psi^{\#})$ where $\psi : X \to Y$ is a continuous map from X to Y viewed as topological spaces, and $\psi^{\#} : \mathcal{O}_Y \to \psi_* \mathcal{O}_X$ is a map of sheaves such that the following compatibility condition is satisfied:

For any point $p \in X$ and any neighbourhood U of $\psi(p) \in Y$, a section $f \in \mathcal{O}_Y(U)$ vanishes as $\psi(p)$ if and only if the section $\psi^{\#} f$ of $\psi_* \mathcal{O}_X(U) = \mathcal{O}_X(\psi^{-1}U)$ vanishes at p.

Whilst the *stalk* $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ is formally defined via a direct limit of groups $\mathcal{F}(U)$ over all open neighbourhoods $x \in U \subset X$ [4, I], for our purposes we shall define this concept at a lower level as follows:

Definition 2.8. Given a sheaf $\mathcal{S}$ on a topological space X, the *stalk* at $p \in X$, denoted $\mathcal{S}_p$, is the ring of germs of sections at p.

Piecing together *Definitions 2.7* and *2.8*, we obtain a reformulation of *2.6*:

Definition 2.9. Let $\mathcal{X}, \mathcal{Y}$ be schemes. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is called *flat* if for all points $p \in X$ the induced map on stalks

$$f^{\#} : \mathcal{O}_{Y,f(p)} \to f_* \mathcal{O}_{X,p}$$

is flat, i.e. $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(p)}$-module.

The concepts here should not be so foreign to us; after all, the modern approach to algebraic geometry views *schemes* as a generalisation of *varieties*. This generalisation is well founded in this instance by the following remark:

Remark 2.10. From *Definition 2.9* we can recover *Definition 2.6*; that is, a *flat morphism of schemes* is a generalisation of a *flat morphism of varieties*.

The crux of the argument is the following theorem:

Theorem 2.11. [5, II Prop. 2.6] *Let k be an algebraically closed field. There is a natural fully faithful functor*

$$t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k), \quad (X, \mathcal{O}_X) \mapsto (\mathrm{Spec}(k[X]), \mathcal{O}_{\mathrm{Spec}(k[X])}),$$

where $\mathfrak{Var}(k)$ is the category of varieties over k, $\mathfrak{Sch}(k)$ is the category of schemes over k, and $(X, \mathcal{O}_X)$ is a variety with a structure sheaf of regular functions.

With this in mind, re-examining *Definition 2.6* we see this is *Definition 2.9* noticing $\mathcal{O}_{\mathrm{Spec}(k[X])} = k[X]$[2]. $\Diamond$

Jumping off this idea, we preform a quick sanity check:

Lemma 2.12. *A map of rings $\psi : A \to B$ is flat if and only if the corresponding spectra map $\Psi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat.*

Proof. Suppose $\psi : A \to B$ is flat. Given $p \in \mathrm{Spec}(B)$, consider

$$\Psi^{\#} : \mathcal{O}_{\mathrm{Spec}(A),\Psi(p)} \to \Psi_* \mathcal{O}_{\mathrm{Spec}(B),p}.$$

By definition of $\mathcal{O}_{\mathrm{Spec}}$ and $\Psi(p)$, this becomes $\Psi^{\#} : A_{\psi^{-1}(p)} \to B_p$. As f is flat, B is a flat A-module, and by extending this to the localisation we see B_p is a flat $A_{\psi^{-1}(p)}$-module, making $\Psi^{\#}$ flat. Hence the induced map from Ψ on stalks is flat, meaning Ψ is flat.

Conversely suppose $\Psi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ hence $\Psi^{\#} : A_{\psi^{-1}(p)} \to B_p$ is flat. As for all $p \in \mathrm{Spec}(B)$, $A_{\psi^{-1}(p)}$ is a flat A-module, hence $A \to A_{\psi^{-1}(p)}$ is a flat map. By transitivity (*Lemma 2.13 (2)*) we deduce $A \to B_p$ is flat for all $p \in \mathrm{Spec}(B)$. We conclude $A \to B$ is flat, as desired. ■

Another few 'natural' algebraic facts that will useful in our understanding are below.

Lemma 2.13.

(1) ***Base Extension:*** *If M is a flat R-module and $R \to S$ is a homomorphism then $S \otimes_R M$ is a flat S-module.*
(2) ***Transitivity:*** *If N is a flat R-module, and M is a flat N-module, then M is a flat R-module.*
(3) ***Localisation:*** *If M is an R-module, then M is flat over R if and only if for all prime ideals $\wp \in R$, the localisation $M_\wp$ is flat over $R_\wp$.*

Proof. While these properties are mentioned in the major texts [4, 5, 9] they are proven in [2, 8]. Hartshorne and Mumford [5, 9] go so far as to translate these properties into "geometric terms" concerning flat morphisms, which have the same properties as above. ■

As we have covered the basic algebraic notions, we are now ready for some examples.

3. Examples & Non-Examples

We begin with specifying a type of subscheme:

Definition 3.1. Let $\mathcal{X} = \mathrm{Spec}\, R$ be an affine scheme. A *closed subscheme* of $\mathcal{X}$ is a scheme $\mathcal{Y}$ that is the spectrum of a quotient of R. (Thus the closed subschemes $\mathcal{Y}$ of $\mathcal{X}$ are in 1-to-1 correspondence to ideals I in the ring R, as $\mathcal{Y} = \mathrm{Spec}\, R/I$.)

[2]Also we must consider the fact we are working with projective varieties in *2.6*, hence we use $k[\widehat{X}]$ as the coordinate ring, and elements $p \in \mathrm{Spec}\, k[X]$ correspond to prime ideals $\wp \subset k[X]$ which (almost!) correspond exactly to elements of X.

This definition can be extended to general schemes, replacing the ideal I with a sheaf of ideals $\mathscr{I}$. In general we can identify closed subschemes with topologically closed subsets of our original scheme, together with a compatibility condition on its sheaf of rings.

Definition 3.2. A *family of schemes* is a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of schemes. The individual schemes in the family are simply the fibers of φ over points of $\mathcal{Y}$.

We shall state the next definition in terms of projective space and projective schemes in preparation for §4 however the analogous definitions can be made for affine space, and used in *Example 3.4*.

Definition 3.3. [4, III.2] A family of closed subschemes of $\mathbb{P}^r$ over the base $S = \operatorname{Spec} A$ is a closed subscheme of $\mathbb{P}^r_A := \operatorname{Proj} A[x_0, \dots, x_r]$.

As there is a canonical morphism $\mathbb{P}^r_A = \operatorname{Proj} A[x_0, \dots, x_r] \to \operatorname{Spec} A$ given by $A \to \mathcal{O}_{\mathbb{P}^r_A}(\mathbb{P}^r_A)$ "=" $A[x_0, \dots, x_r]$, the morphism $\mathcal{X} \to \operatorname{Spec} A$ naturally given by restriction for $\mathcal{X} \subset \mathbb{P}^r_A$ allows us to view $\mathcal{X}$ as a family over the base $\operatorname{Spec} A$ via the fibers.

Our first example of a flat family appears in Hartshorne [5, III.9]:

Example 3.4. Let $\mathcal{X} \subset \mathbb{P}^3$ be the twisted cubic curve; a closed subscheme of $\mathbb{P}^3$. The projection map

$$\mathbb{P}^3 \supset \mathcal{X} \xrightarrow{\pi} \pi(\mathcal{X}) \subset \mathbb{P}^2, \qquad \pi(\mathcal{X}) \text{ a nodal cubic curve in } \mathbb{P}^2,$$

will be useful in our analysis of $\mathcal{X}$.

(See the image to the right for a rough picture). We shall show that the family $\{\mathcal{X}_a\}_{a \in \mathbb{A} \setminus \{0\}}$ given by $\sigma_a(\mathcal{X}) = \mathcal{X}_a$ (for $a \in \mathbb{A} \setminus \{0\}$),

$$\sigma_a : [x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1} : a x_n],$$

forms a flat family parameterized by $\mathbb{A} \setminus \{0\}$, where $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_0 = \pi(\mathcal{X})$ (as sets). In fact the fiber at 0, $\mathcal{X}_0$, consists of the nodal cubic $\pi(\mathcal{X})$ together with some nilpotent elements at the double point.

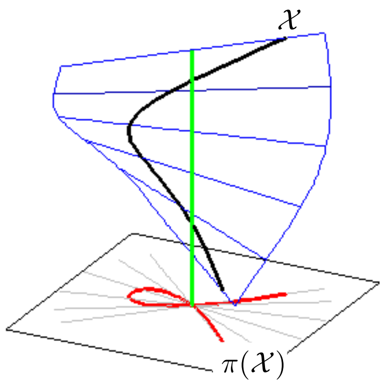


FIGURE 1. Image: [1].

Take affine patches of $\mathbb{P}^3, \mathbb{P}^2$. Since we only care about the behaviour of $\{\mathcal{X}_a\}_{a \in \mathbb{A} \setminus \{0\}}$ near the origin, we will from this point forth work solely in $\mathbb{A}^3, \mathbb{A}^2$. $\mathcal{X}$ is given by the parametric equations

$$x = t^2 - 1,$$
$$y = t^3 - t,$$
$$z = t,$$

in the affine coordinates on $\mathbb{A}^3$. For any $a \neq 0$, consider $\mathcal{X}_a$ given by the equations

$$x = t^2 - 1,$$
$$y = t^3 - t,$$
$$z = at.$$

$\{\mathcal{X}_a\}_{a\in\mathbb{A}\setminus\{0\}}$ is indeed a family of schemes as the whole family is isomorphic to $\mathcal{X}_1 \times (\mathbb{A} \setminus 0)$ which by the canonical projection map ensures each $\mathcal{X}_a$ is indeed a fiber and the family is flat; the dimension of each fiber is the same due to the scheme-isomorphism $\mathcal{X}_a \cong \mathcal{X}_1$, hence the Hilbert functions agree and by *Definition A* the family is flat.

As $\mathcal{X} \subset \mathbb{A}^3_{\mathbb{A}\setminus\{0\}}$ is a closed scheme which is flat over $\mathbb{A} \setminus \{0\}$, it has a well defined scheme-theoretic closure which remains flat [5, III Prop. 9.8]. Let $\bar{\mathcal{X}}$ be this closure, called the *total family extended over* $\mathbb{A}$. As this is a closed subscheme, it has an ideal $I \subset k[a, x, y, z]$ associated to it. We find

$$I = (a^2(x + 1) - z^2, a^3y + a^2z - z^3, xz - ay, y^2 - x^2(x + 1)).$$

As we are interested in the family at $a = 0$, $\mathcal{X}_0$ is given by the ideal

$$I_0 = I_{a=0} = (z^2, z^3, xz, y^2 - x^2(x + 1)).$$

Set-theoretically, $\mathcal{X}_0$ should agree with the projection of $\mathcal{X}_1$, which is the nodal cubic $y^2 = x^2(x + 1)$. From I_0, we see in fact $\mathcal{X}_0$ is a scheme with the same underlying space as $\pi(\mathcal{X}_1)$, but in the local ring $\mathcal{O}_{\mathcal{X}_0,(0,0,0)} = k[x, y, z]/I_0$ we have a nilpotent element z of degree 2. According to Hartshorne, "[i]t seems as if the scheme $\mathcal{X}_0$ is retaining the information that it is a limit of a family of space curves, by having these nilpotent elements which point out of the plane". See *Figure 2* for an illustration of this example. ♢

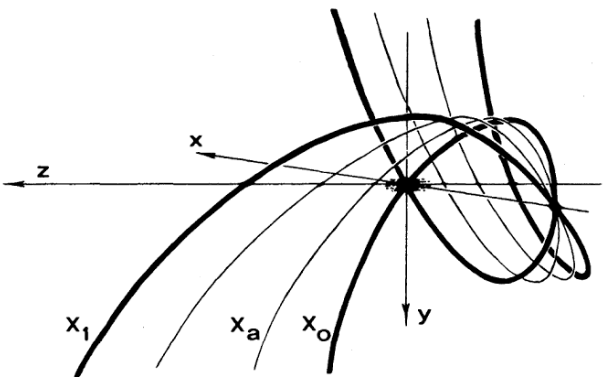


FIGURE 2. The flat family $\mathcal{X}_a$ of subschemes of $\mathbb{P}^3$. Image: [5].

Our first non-example is a bit of a cheat; we know from the beginning it will not be a flat map for dimension reasons.

Non-example 3.5. [10, 5, I §4] The blow-up of $\mathbb{A}^2$ at the origin O is defined as:

$$\begin{aligned} B_0\mathbb{A}^2 &= \{\text{Any line through } O \text{ in } \mathbb{A}^2\text{, together with any choice of point on the line}\} \\ &= \mathbb{V}(xu - yt) \subset \mathbb{A}^2 \times \mathbb{P} \quad \text{where the affine coordinates are } (x, y) \\ &\quad\; \text{and the projective coordinates are } [t : u]. \end{aligned}$$

This also leaves us with a map $\pi : B_0\mathbb{A}^2 \to \mathbb{A}$ that is the restriction to $B_0\mathbb{A}^2$ of the projection map $\mathbb{A}^2 \times \mathbb{P} \to \mathbb{A}^2$.

From this we define the blow-up of a variety Y (also at O) to be $\widetilde{Y} = \overline{\pi^{-1}(Y \setminus \{O\})}$ (where $\overline{\cdots}$ is the closure inside $\mathbb{A}^2 \times \mathbb{P}$).

Take for example $Y = \mathbb{V}(y^2 - x^2(x+1))$. The blow-up of Y at O is given in *Figure 3*.

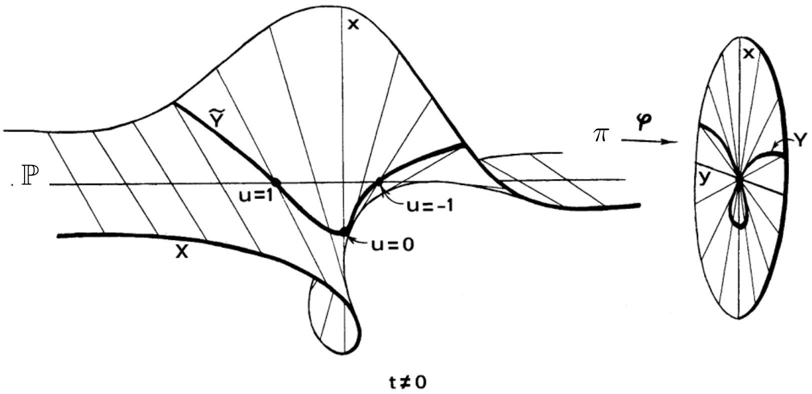


FIGURE 3. $\widetilde{Y}$ is defined by $u^2 = x + 1$, $y = xu$. Image: [5].

If $p \in Y \subset \mathbb{A}^2$, $p \neq O$, then $\pi^{-1}(p)$ consists of just one point; moreover $\pi : B_0Y \to Y$ is an isomorphism on the complement of $\pi^{-1}(O)$. However at $p = O$, the preimage jumps from being one point to $\pi^{-1}(O) \cong \mathbb{P}$. The dimension of $\pi^{-1}(p)$ jumps at $p = O$, hence the *degree* of the Hilbert polynomial also jumps at this point. Thus the fibers of π at $p \in Y$ definitely do not have the same Hilbert polynomial. We conclude the family $\pi : B_0Y \to Y$ is not flat[3]. ♢

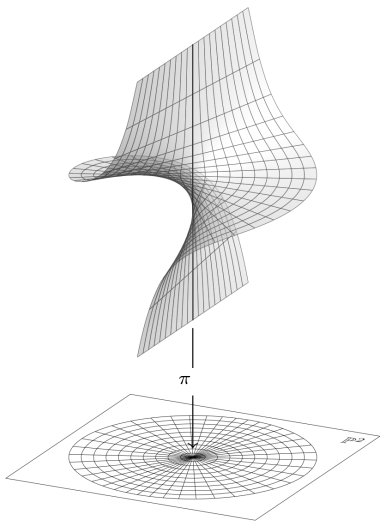


FIGURE 4. Image: [12].

Remark 3.6. In fact, if the purpose of *Non-example 3.5* was to demonstrate the blow-up map is not flat, we do not need the variety $\mathbb{V}(y^2 - x^2(x+1))$; looking at the blow-up of the entire space at O

$$\pi : B_0\mathbb{A}^2 \to \mathbb{A}^2,$$

by the same calculation as in *Non-example 3.5* we see the dimension of the fibers surges at O (*left*). The purpose of *Non-example 3.5* is to serve in contrast to *Example 3.4*; the latter is a case where the twisted cubic is part of a flat family, and the former a case where the twisted cubic (somehow given by blowing-up $y^2 = x^2(x+1)$) is not part of a flat family. The key take-away here is that in a flat family, it is not the inherent *space* that makes it flat, rather the *map* from the space to the 'parameter space' of the fibers. ♢

To understand our next few examples we need to introduce the notion of *finiteness*.

[3]As is perhaps expected when dealing with a construction known as *blowing-up*.

Definition 3.7. [4, III.1.1] A morphism of schemes $\varphi : \mathcal{X} \to \mathcal{Y}$ is called *finite* if for every $y \in \mathcal{Y}$ there is an open affine neighbourhood $y \in V = \operatorname{Spec} B \subset \mathcal{Y}$ such that $\varphi^{-1}(V) = \operatorname{Spec} A$ is itself affine, and if via the pullback map

$$\varphi|_V^{\#} : B = \mathcal{O}_Y(V) \to A = \mathcal{O}_X(\varphi^{-1}V),$$

A is a *finitely generated* B-module.

According to Eisenbud & Harris [4, III.1], this is a "very stringent condition" that forces all the fibers of such a morphism to be finite.

We have not yet defined what a *fiber* of a scheme morphism actually is, instead relying on the reader's intuition regarding fibers in §3. Let us correct that now.

Definition 3.8. If $\varphi : \mathcal{X} \to \mathcal{Y}$ is a morphism of schemes, the *fiber* of $y \in \mathcal{Y}$ is the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \Bbbk(y)$, where $\Bbbk(y)$ is the *residue field of the point* y, defined as $A_\wp/\wp \cdot A_\wp$ where $U = \operatorname{Spec} A$ is an affine neighbourhood of $y \in Y$ and $\wp$ is the prime ideal of A corresponding to y.

This more general definition is to account for the generic points of a scheme; a non-closed point might not have a well defined preimage.

One immediate consequence of these definitions is that it provides us with our first theorem in connection to dimension:

Theorem 3.9. (Precursor to the *HP-Flatness Theorem*). *Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a finite morphism of affine schemes. Assume $\mathcal{Y}$ is Noetherian, reduced and irreducible*[4]*. Then φ is flat if and only if the integer*

$$\dim_{\Bbbk(y)}[\varphi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Bbbk(y)]$$

is independent of $y \in Y$.

Moreover, if $A_y = \varphi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Bbbk(y)$ then $\operatorname{Spec} A_y$ *is the fiber of f over y.*

Proof. See [9, III.10]. ■

Example 3.10. [9, III.10 Ex. Q] Let k be an algebraically closed field and define

$$f : \mathbb{A} \to \mathbb{A}, \qquad x \mapsto x^2,$$

over k. Viewing $\mathbb{A}$ as a scheme we see it is indeed Noetherian, reduced and irreducible (all in the scheme-theoretic sense). Note that for any $y \in \mathbb{A}$, $\mathbb{A}$ itself is an open affine neighbourhood, hence the pullback map

$$f^{\#} : k[x] \to k[x], \qquad x \mapsto x^2,$$

trivially leaves $k[x]$ a finitely generated $k[x]$-module. We are thus in a position to apply *Theorem 3.9.*

Given $a \in \mathbb{A}$, the fiber of f over $x = a$ is simply $\operatorname{Spec}(k[x]/(x^2 - a))$ via [10, §6.4]. As $\dim_k k[x]/(x^2 - a) = 2$ for any a, and the residue field at a point is just k, we conclude

$$\dim_{\Bbbk(y)}[\varphi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Bbbk(y)] =^5 \dim_k k[x]/(x^2 - a) = 2,$$

hence f is flat. ◊

[4]Terms all clarified in [4, I.2]

[5][4, I.3 Exercise I-46 (g)]

Non-example 3.11. [9, III.10 Ex. R] Let k be an algebraically closed field and define

$$v_2 : \mathbb{A}^2 \to \mathcal{Y} = \mathbb{V}(x_1x_3 - x_2^2) \subset \mathbb{A}^3, \qquad (x, y) \mapsto (x^2, xy, y^2),$$

the affine two-dimensional Veronese map. The variety $\mathbb{V}(x_1x_3 - x_2^2)$ (viewed as a scheme) is Noetherian, reduced and irreducible. The pullback map

$$v_2^{\#} : k[x, y, z]/(xz - y^2) \to k[x, y], \qquad g(x, y, z) \mapsto g(x^2, xy, y^2),$$

leaves $k[x, y]$ a finitely generated $k[x, y, z]/(xz - y^2)$-module indeed.

If $p = (a, b, c) \in \mathcal{Y}$ the fiber of v_2 over p is then given by[6]

$$\begin{array}{ccc} \operatorname{Spec}(k[x,y]) \otimes_{\operatorname{Spec}(k[x,y,z]/(xz-y^2))} \operatorname{Spec}(\Bbbk(p)) & \overset{\pi_1}{\dashrightarrow} & \operatorname{Spec} \Bbbk(p) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle i} \\ \mathbb{A}^2 \text{``=''} \operatorname{Spec}(k[x,y]) & \xrightarrow{v_2} & \mathcal{Y} = \mathbb{V}(xz - y^2) \text{``=''} \operatorname{Spec}(k[x,y,z]/(xz-y^2)) \end{array}$$

By definition,

$$\operatorname{Spec} A \otimes_{\operatorname{Spec} B} \operatorname{Spec} C = \operatorname{Spec}(A \otimes_B C),$$

therefore we obtain:

$$\begin{aligned} &\operatorname{Spec}(k[x, y]) \otimes_{\operatorname{Spec}(k[x,y,z]/(xz-y^2))} \operatorname{Spec}(\Bbbk(p)) \\ &= \operatorname{Spec}(k[x, y] \otimes_{k[x,y,z]/(xz-y^2)} \Bbbk(p)) \\ &= \operatorname{Spec}(k[x, y]/(x^2 - a, xy - b, y^2 - c)), \end{aligned}$$

via $v_2(x, y) = (x^2, xy, y^2)$. So by this sketch the fiber of v_2 over p is

$$\operatorname{Spec}(k[x, y]/(x^2 - a, xy - b, y^2 - c)).$$

If $a = c = 0$ necessarily $b = 0$ so the fiber becomes $\operatorname{Spec}(k[x, y]/(x^2, xy, y^2))$ and

$$\dim_{\Bbbk((0,0,0))}[v_{2*}\mathcal{O}_{\mathbb{A}^2} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Bbbk((0, 0, 0))] = \dim_k k[x, y]/(x^2, xy, y^2) = 3,$$

However if $a \neq 0$ and $b = c = 0$, then

$$\dim_{\Bbbk((a,0,0))}[v_{2*}\mathcal{O}_{\mathbb{A}^2} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Bbbk((a, 0, 0))] = \dim_k k[x, y]/(x^2 - a, xy, y^2) = 2,$$

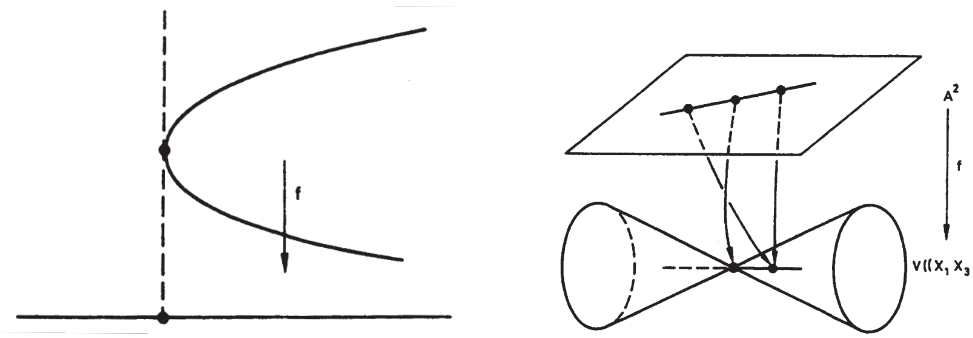
as k is algebraically closed.

Therefore by *Theorem 3.9* we conclude v_2 is not flat. ♢

Mumford [9] includes illustrations for these two morphisms which give an indication of the origin of the term 'flat':

[6] We should technically be writing things of the form $\mathbb{A}^2 = \operatorname{Specm}(k[x, y])$ however it is generally the convention not to worry about the generic points and work with Spec instead.

Figure 5



(A) *Example 3.10.* (B) *Non-example 3.11.*

As has been indicated by this example and non-example pair, the geometric notion of *dimension* is very important to flatness. Indeed our first definition of flatness, *Definition A*, was based on the idea of the Hilbert polynomial, the degree of the leading term of which is the dimension of the space we are considering. We shall now reconcile *Definition B* with *Definition A*, and the reader is invited to consult [4, 5, 7, 9, 11] should more examples of flat families be desired.

4. An Invariant Of Hilbert's

Recall *Definitions 3.1 - 3.3* regarding *subschemes*. We are now in a position and sufficiently motivated to give the main theorem of the essay.

Theorem 4.1. The HP-Flatness Theorem. *A family $\mathcal{X} \subset \mathbb{P}^r_B$ of closed subschemes of a projective space over a reduced connected base B is flat if and only if all fibers have the same Hilbert polynomial.*

Proof. We shall prove this theorem for the case $B = \operatorname{Spec} K[t]_{(t)}$, where K is a field, following [4, Prop. III-56].

As *Definition 3.1* indicates, the closed subscheme $\mathcal{X}$ is given by an ideal I of $K[t]_{(t)}[x_0, \dots, x_r]$ which for our projective case must be homogeneous in $x_0, \dots, x_r$. Note that each graded piece

$$R_m = (K[t]_{(t)}[x_0, \dots, x_r]/I)_m = K[t]_{(t)}[x_0, \dots, x_r]/I_m$$

is a (finitely generated) $K[t]_{(t)}$-module.

From *Lemma 2.4*, the family $\mathcal{X} \to B$ is flat if and only if the local ring $\mathcal{O}_{X,p}$ is $K[t]_{(t)}$-torsion-free for each point $p \in X$. If this is the case, then the torsion submodule of $R = K[t]_{(t)}[x_0, \dots, x_r]/I$, denoted T_R, vanishes if we invert any x_i:

If $g \in R$ is an element of T_R, then $f(t)g = 0$ for some $f(t) \in K[t]_{(t)}$. Then for some point $x \in X$, g and some open neighbourhood $U \ni x$ are members of $\mathcal{O}_{X,x}$. Restricting to a basic open set D_{x_i}, we see $\mathcal{O}_X(D_{x_i}) = R_{x_i}$. So by inverting x_i in R, we recover $g \in \mathcal{O}_{X,x}$ and as $\mathcal{O}_{X,x}$ is $K[t]_{(t)}$-torsion-free at x, we conclude $g = 0$, hence the torsion submodule of R vanishes if we invert x_i.

Therefore T_R is killed by some power of $I_+ = (x_0, \dots, x_r)$ and thus T_R meets only finitely many R_m, that is to say meets only finitely many graded components of R.

By *Lemma 2.3*, over a principal ideal domain finitely generated modules are torsion free if and only if they are free. If R_m is a graded component of R, as $K[t]_{(t)}$ is a principal ideal domain and R_m is finitely generated as a $K[t]_{(t)}$-module, it is torsion free if and only if it is free. As we have just demonstrated, it is certainly torsion free for all but finitely many m. Furthermore, R_m is free if the number of its generators:

$$\dim_K R_m/(t)R_m = \dim_K R_m \otimes_{K[t]_{(t)}} K \text{ by Nakayama's Lemma,}$$

(as K is the residue field) is equal to its rank:

$$\dim_{K(t)} R_m \otimes_{K[t]_{(t)}} K(t) \text{ by definition of rank [8],}$$

where the field of fractions of $K[t]_{(t)} = K(t)$.

A quick calculation of the fibers of $\mathcal{X} \to B$ over the two points of $\operatorname{Spec} K[t]_{(t)} = \{(0), (t)\}$ gives us

$$\begin{aligned} \text{The fiber over } (0): \ X_{(0)} &= \operatorname{Spec}(R \otimes_{K[t]_{(t)}} K), \\ \text{The fiber over } (t): \ X_{(t)} &= \operatorname{Spec}(R \otimes_{K[t]_{(t)}} K(t)), \end{aligned}$$

hence

$$h_{X_{(0)}}(m) = \dim_K R_m \otimes_{K[t]_{(t)}} K = \dim_{K(t)} R_m \otimes_{K[t]_{(t)}} K(t) = h_{X_{(t)}}(m).$$

By our argument above these Hilbert functions agree for all but at most finitely many m, thus for m_0 large enough

$$h_{X_{(0)}}(m) = h_{X_{(t)}}(m) \quad \text{for all } m \geq m_0.$$

Therefore all fibers of B have the same Hilbert polynomial, as required. ■

Returning to the simpler language of varieties (via *Theorem 2.11*), a corollary of this theorem is the following:

Corollary 4.2. *The Hilbert polynomial is constant across the fibers in a flat family $X \to Y$ of projective subvarieties parameterized by a connected base Y.* ■

Remark 4.3. As a consequence of the *HP-Flatness Theorem* and *Corollary 4.2*, *Definition A* and *Definition B* are equivalent. ◊

We can intuitively see why *Example 3.10* was flat while *Non-example 3.11* was not. In fact, *Non-example 3.11* gives an explicit computation of the Hilbert function and demonstrates it is not constant over the fibers, while *Example 3.10* calculates the Hilbert function to take value 2 for any fiber, and is hence constant.

A more general proof of the *HP-Flatness Theorem* can be found in [11, 4.3] or [5, III.9] (in particular, Hartshorne produces a homological algebraic proof).

5. The Answer To Many Prayers

> *"I call our world Flatland, not because we call it so, but to make its nature clearer to you, my happy readers, who are privileged to live in Space."*
> - Edwin A. Abbott, *Flatland: A Romance of Many Dimensions*

In the 1880's Edwin Abbott wrote *Flatland* - the tale of a square who attempts to understand and share the existence of three dimensions while living in a strict, classist society set in a two dimensional plane - as both a commentary on his current Victorian setting and also as a means of playing with our conception of *dimension*. Just as dimension is key in the minds of Flatlanders in an attempt to understand their reality, so too is dimension a key concept in mathematics as an attempt to understand the geometry of structures. In the previous section we proved our initial definitions of *flatness* agreed with each other; the algebraic formulation over modules agrees with the geometric notion that the Hilbert polynomial of the fibers remains the same. We now turn to the application of flatness in scheme theory: allowing the idea of a 'continuously varying' family of schemes.

As it stands, the definition of a family of schemes is quite general: it is a morphism of schemes, where 'family' is interpreted as the collection of fibers under this morphism. If we wish to determine how the fibers relate to each other, what they have in common and how that varies across the parameter space, we must first introduce the counterpart of continuity: limits.

Let B be a non-singular one-dimensional scheme - e.g. $\operatorname{Spec} K[t]_{(t)}$ from the *HP-Flatness Theorem*. Let $0 \in B$ be some closed[7] point and define $B^* = B \setminus \{0\}$. Consider (as in the *HP-Flatness Theorem* but now for affine space) a closed subscheme $\mathcal{X} \subset \mathbb{A}^n_{B^*}$, which again is a family $X_b = \pi^{-1}(b)$ of closed subschemes parameterised by B^* given by the fibers of $\pi : \mathcal{X} \to B^*$. To get a sense of how continuously these fibers vary, we ask for the limit of the schemes X_b as $b \to 0$. To answer this we take the closure $\overline{\mathcal{X}} \subset \mathbb{A}^n_B$ of $\mathcal{X}$ in $\mathbb{A}^n_B$[8] and define

$$\lim_{b\to 0} X_b = \overline{X}_0 = \text{ fiber of } \overline{\mathcal{X}} \text{ over the point } 0 \in B.$$

Obviously if we want the fibers of our family to relate to each other in the smooth, continuous fashion we wish for, in a general family of schemes $\pi : \mathcal{X} \to B$, the limit $\lim_{b\to 0} X_b$ should actually equal the fiber $X_0 = \pi^{-1}(0)$. In a flat family, we have exactly that.

Theorem 5.1. [4, Prop. II-29] *Let $B = \operatorname{Spec} R$ be a non-singular, one dimensional affine scheme with a closed point $0 \in B$. Let $\mathcal{X} \subset \mathbb{A}^n_B$ be any closed subscheme and $\pi : \mathcal{X} \to B$ the standard projection. Then π is flat (over 0) if and only if the fiber X_0 is the limit of the fibers X_b as $b \to 0$.* ■

Recall *Example 3.4*: this is a nice example of the theorem in action. Taking the limit of $\mathcal{X}_a$ as $a \to 0$ we could obtain the fiber $\mathcal{X}_0$ as we computed it explicitly. The isomorphism $\mathcal{X}_a \cong \mathcal{X}_1$ mentioned in *Example 3.4* is also evidence that

[7] The closure of a point p is $\{q : q \supset p\} \subset \operatorname{Spec} B$. Hence p is closed if and only if it is maximal.

[8] Note B^* has become B.

the family $\{\mathcal{X}_a\}_{a\in\mathbb{A}}$ varies continuously and the fibers 'look like' one another. In short: if some mathematician prayed for a condition that determines when families of schemes vary continuously, flatness is the answer to that prayer.

To conclude this essay, we shall make mention of the connection of flatness to topology. Rather than being an involved connection concerning topological objects or maps, and the flatness of them, we instead are going to draw a parallel between the two subjects in an effort to vaguely indicate our notion of *flat* is already present elsewhere in the topography of mathematics.

Definition 5.2. A fiber bundle is a tuple (E, B, π, F) where E, B, F are topological spaces, B is connected and $\pi : E \to B$ is a continuous surjection such that for any point $x \in E$, there exists an open neighbourhood $\pi(x) \in U \subset B$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \overset{\phi}{\dashrightarrow} & U \times F \\ {\scriptstyle\pi}\downarrow & \swarrow_{\pi_1} & \\ U & & \end{array}$$

where ϕ is a homeomorphism and π_1 is the canonical projection map.

Notice the similarity between this definition and *Definition 3.3* of a family of closed subschemes. If we are trying to make the point that a flat family and a fiber bundle are analogous in definition, we should have some result that reinforces this idea. Thankfully we do, due to Grothendieck:

Theorem 5.3. Generic Flatness Theorem. [3, §14.2] *If $\pi : \mathcal{X} \to B$ is a reasonable*[9] *family of schemes over a reduced base, then there is an open dense subset $U \subset B$ such that the restricted family $\pi^{-1}U \to U$ is flat.* ■

This we see is similar to the topological result:

Theorem 5.4. *If $f : M \to N$ is a differentiable map of compact C^∞ manifolds, then there is a dense collection of open subsets $U \subset N$ such that the restriction $f|_{f^{-1}(U)}$ is a fiber bundle.* ■

We conclude that although flat morphisms come from a heavy algebraic background, through examples and the *HP-Flatness Theorem* we see they have a strong connection to uniformity in dimension and by the *Generic Flatness Theorem* are a frequent and fundamental feature of families of schemes.

In short: our geometrical expectations do not fall flat.

[9] Explained further in [3].

References

[1] Arapura, D. *Figure 1*, `www.math.purdue.edu/~dvb/graph/nodal.html`, (accessed 2017).
[2] Bourbaki, N. *Commutative Algebra Chapters 1-7*, 2nd ed. Springer-Verlag, (1989).
[3] Eisenbud, D. *Commutative Algebra with a View Toward Algebraic Geometry*, Springer GTM 150, (1995).
[4] Eisenbud, D. & Harris, J. *The Geometry of Schemes*, Springer GTM 197, (1999).
[5] Hartshorne, R. *Algebraic Geometry*, Springer GTM 52, (1977).
[6] Lang, S. *Algebra*, 3rd ed. Addison-Wesley, (1993).
[7] Lehmann, B. *Flatness*, expository paper, available at `www2.bc.edu/brian-lehmann`, (accessed 2017).
[8] Matsumura, H. (M. Reid, Trans.) *Commutative Ring Theory*, Cambridge University Press, (1987).
[9] Mumford, D. *The Red Book of Varieties and Schemes*, 1st ed. LNM 1358, Springer-Verlag, (1988).
[10] Ritter, A. *C3.4 Algebraic Geometry class notes*, the University of Oxford, available at `courses.maths.ox.ac.uk/node/view_material/4672`, (2017).
[11] Shafarevich, I. *Basic Algebraic Geometry: Volume 2*, 2nd ed. Springer-Verlag, (1997).
[12] Staats, C. *Figure 4*, available at `math.uchicago.edu/~cstaats/blowup-one-pt.pdf`, (accessed 2017).
[13] Weibel, C. A. *An Introduction to Homological Algebra*, Cambridge University Press, (1994).