Kreyszig’s textbook is a suitable guide for this part of the module.

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1 Terminology

• A differential equation (DE) is an equation involving functions of a set of variables and their derivatives.

• A solution of a differential equation is a set of functional expressions for the variables that reduce the differential equation to an identity.

• Ordinary differential equations (ODE) involve only ordinary derivatives.

• Partial differential equations (PDE) involve partial derivatives.

• We distinguish independent and dependent variables in a differential equation: the dependent variables are differentiated with respect to the independent variables.

• The order of a differential equation is the order of the highest derivative.
A differential equation is **linear** if it can be written so that it is linear in the dependent variables and their derivatives.

A **linear** differential equation is **homogeneous** in a dependent variable $f$ if it can be written in a way such that each term involves $f$ or one of its derivatives.

A **general** solution to a differential equation contains undetermined constants. A **particular** solution has none. The **general** solution refers to the most general solution. The undetermined constants can be thought of as arising from constants of integration.

**Initial conditions** or **boundary conditions** are a set of specific values of the dependent variable that are used to restrict general solutions to particular solutions.

We use various variables in our examples of differential equations. In this module, we use a prime symbol ($'$) to refer to the differential operator $\frac{d}{dx}$ and the dot above ($\dot{}$) to refer to $\frac{d}{dt}$.

For simplicity, in the remainder of these notes, we will exclusively use $y$ as the dependent variable and $x$ as the independent variable in the original differential equations, and express differentiation with respect to $x$ by the prime symbol.

We typically write integration constants explicitly, so that indefinite integrals may be substituted with any version of the antiderivative. The constants are either left undetermined in the general solution, or fixed by initial or boundary values in the very last step.

These notes summarize strategies for finding general solutions, since the principle of fixing constants with initial or boundary values is the same in every case.

Tip: solutions to differential equations are easy to check, by substituting them into the original equation. Do so whenever time allows.

## 2 First order separable

An ODE of the form $y' = f(x)$ can be solved by direct integration.

More generally, an ODE is **separable** if it can be written in the form

$$g(y) \cdot y' = f(x).$$

To solve this equation, integrate both sides with respect to $x$. On the left-hand side, make a simple change of variables in order to integrate over $y$, using $y' dx = dy$ under the integral sign. Then we have

$$\int g(y) \, dy = \int f(x) \, dx + C.$$

Integrate both sides, and then solve for $y$ in terms of $x$.

### 2.1 Separable after a change of variables

Some ODEs are not directly separable, but can be brought into a separable form with a change of variables. We have seen two classes of examples of this type of ODE.

For an equation of the form

$$y' = f \left( \frac{y}{x} \right),$$

substitute $u = y/x$ as the dependent variable.

For an equation of the form

$$y' = f \left( ax + by + c \right),$$

where $a, b, c$ are constants, substitute $u = ax + by + c$ as the dependent variable.

## 3 First order linear

A first-order linear ODE can be brought into the following **standard** form:

$$y' + p(x)y = r(x).$$
3.1 Homogeneous

If \( r(x) = 0 \) in eq. (1), then the equation is homogeneous. It is then also separable and can be solved as above.

Alternatively, we introduce the integrating factor \( e^{\int p(x) \, dx} \). Then,

- Multiply the equation through by the integrating factor.
- Recognize the left-hand side as the total derivative \( \frac{d}{dx} \left( ye^{\int p(x) \, dx} \right) \). This is also the point at which to check that the integrating factor was correct, or to rederive the formula for the integrating factor.
- Integrate the equation (including an integration constant).
- Solve for \( y \) in terms of \( x \).

3.2 Nonhomogeneous

Use the same integrating factor method outlined above. The difference is that the right-hand side will be a nonzero function of \( x \).

3.3 Linear after a change of variables

Some ODEs are not directly linear, but can be brought into a linear form with a change of variables. One example of this type is the Bernoulli equation,

\[
y' + p(x)y = g(x)y^a,
\]

where \( a \in \mathbb{R} \), and the equation is nonlinear if \( a \neq 0 \) and \( a \neq 1 \). Here, the recommended change of variables is \( u = y^{1-a} \).

4 How to solve linear ODE

A linear ODE of order \( n \) can be brought into the following standard form:

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x),
\]

where \( y^{(k)} \) denotes the \( k \)-th derivative of \( y \). It is important that there is no coefficient in front of \( y^{(n)} \). (If there is, divide the entire equation through by that coefficient to get the standard form.)

The equation is homogeneous if \( r(x) = 0 \) in the standard form, and nonhomogeneous otherwise. Every nonhomogeneous linear ODE has a homogeneous counterpart,

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,
\]

which is called its complementary equation. Solutions to the complementary equation are called complementary solutions.

The solutions to a homogeneous linear ODE form a vector space. If \( y_1 \) and \( y_2 \) are solutions to a given ODE, then so is \( c_1y_1 + c_2y_2 \), for any constants \( c_1 \) and \( c_2 \).

The dimension of the vector space of solutions equals the order of the homogeneous ODE.

Thus we outline the following general solution strategies for linear ODE:

- For a homogeneous ODE of order \( n \), find a set of \( n \) linearly independent particular solutions, \( \{y_1, y_2, \ldots, y_n\} \). The general solution is

\[
y = c_1y_1 + \cdots + c_ny_n,
\]

where \( c_1, \ldots, c_n \) are undetermined constants.

- For a nonhomogeneous ODE of order \( n \), solve the complementary equation as above. Find a single particular solution \( y_p \) to the nonhomogeneous equation. Then the general solution is

\[
y = c_1y_1 + \cdots + c_ny_n + y_p.
\]
These strategies depend on being able to find particular solutions. We can’t do this in full generality, but we know how to do it for certain types of equations.

Whenever we look for particular solutions, we are free to ignore constants of integration that arise along the way, or set them to arbitrary values.

In the following sections, we focus on second-order equations \( n = 2 \), for which we rename the coefficients in the standard form as follows:

\[
y'' + p(x)y' + q(x)y = r(x).
\] (6)

Several of the methods described below have straightforward generalizations to higher-order ODE.

5 Second order linear homogeneous

We are seeking two linearly independent particular solutions, \( y_1 \) and \( y_2 \), to use in the strategy of the previous section.

5.1 Reduction of order

If a single (nonzero) particular solution \( y_1 \) is known or can be found, this method constructs a second, linearly independent solution \( y_2 \).

- Write the second solution as 
  \[ y_2(x) = u(x) y_1(x). \]
  
  The aim is to solve for the function \( u \).

- Substitute this expression for \( y_2 \) into the ODE. The result is a first-order homogeneous ODE for \( u' \). (Terms proportional to \( u \) drop out because \( y_1 \) is a particular solution.)

- Solve the ODE for \( u' \) by the methods of section 3.1, and then integrate over \( x \) to obtain \( u \).

- Substitute this solution into the equation \( y_2 = u y_1 \) and then write the general solution as in eq. (4).

5.2 Constant coefficients

Consider an equation of the form

\[
y'' + ay' + by = 0,
\]

where \( a \) and \( b \) are constants.

To solve it, use the **trial solution**

\[ y = e^{\lambda x}, \]

where we assume that \( \lambda \) is a constant. Substitute this expression into the differential equation and divide through by \( e^{\lambda x} \). The result is the **characteristic equation** (sometimes called the **auxiliary equation**) \( \lambda^2 + a\lambda + b = 0 \), which is a quadratic solution with two roots \( \lambda_1, \lambda_2 \) obtained from the quadratic formula.

If \( \lambda_1 \neq \lambda_2 \), then the general solution is

\[ y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \]

If \( \lambda_1 = \lambda_2 \), then we can rename this double root as \( \lambda \), and the general solution is

\[ y = c_1 e^{\lambda x} + c_2 e^{\lambda x}. \]
5.3 Euler-Cauchy

An Euler-Cauchy equation (at 2nd order) is one that can be written in the form

\[ x^2 y'' + a xy' + b y = 0, \]

where \( a \) and \( b \) are constants.

Note that this equation is not written in standard form, since the coefficient of \( y'' \) is \( x^2 \).

To solve it, use the trial solution

\[ y = x^m, \]

where we assume that \( m \) is a constant. Substitute this expression into the differential equation and divide through by \( x^m \). The result is the characteristic/auxiliary equation

\[ m^2 + (a - 1)m + b = 0, \]

which is a quadratic solution with two roots \( m_1, m_2 \) obtained from the quadratic formula.

If \( m_1 \neq m_2 \), then the general solution is

\[ y = c_1 x^{m_1} + c_2 x^{m_2}. \]

If \( m_1 = m_2 \), then we can rename this double root as \( m \), and the general solution is

\[ y = c_1 x^m + c_2 (\log x) x^m. \]

We observe that the change of variables \( x = e^t \) converts the Euler-Cauchy equation to one with constant coefficients, which explains the similarity in the structure of their solutions.

5.4 Existence and uniqueness of solutions, and the Wronskian [omitted in 2018]

Consider a homogeneous linear second-order ODE, which takes the form

\[ y'' + p(x)y' + q(x)y = 0. \] (7)

We assume that \( p(x) \) and \( q(x) \) are continuous on an interval containing the point \( x_0 \). The following statements are valid on that interval.

- If \( p(x) \) and \( q(x) \) are continuous, then there exists a unique solution to eq. (7) with fixed initial values \( y(x_0) = k_0, \ y'(x_0) = k_1 \).
- The Wronskian of two differentiable functions \( f, g \) is defined to be

\[
W(f, g)(x) = \begin{vmatrix} f(x) & f'(x) \\ g(x) & g'(x) \end{vmatrix} = \det \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}
\]

- Solutions \( y_1 \) and \( y_2 \) to eq. (7) are linearly independent if and only if \( W(y_1, y_2) \neq 0 \).
- Abel’s identity: If \( y_1 \) and \( y_2 \) are solutions to eq. (7), then

\[ W(y_1, y_2)(x) = W_0 e^{-\int p(x) \, dx} \]

for some constant \( W_0 \).

- Eq. (7) has a general solution of the form \( c_1 y_1 + c_2 y_2 \), where \( y_1 \) and \( y_2 \) are linearly independent, and \( c_1 \) and \( c_2 \) are undetermined constants.

- The general solution includes all solutions.

6 Second order linear nonhomogeneous

We now seek a single particular solution \( y_p \), to use in the strategy outlined in section 4, combined with the techniques for solving homogeneous equations listed in section 5.
6.1 Undetermined coefficients of trial functions

Depending on the form of \( r(x) \) and the type of complementary equation involved, there are specific trial functions that can be used. Substitute the trial function into the ODE and solve for the constant coefficient(s).

<table>
<thead>
<tr>
<th>( r(x) ) with constant coefficients</th>
<th>Trial Function</th>
<th>Solve for</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ke^{\gamma x} ) (( \gamma \in \mathbb{C} ))</td>
<td>( Ce^{\gamma x} )</td>
<td>( C )</td>
</tr>
<tr>
<td>( kx^n ) (( n \in \mathbb{Z}, n \geq 0 ))</td>
<td>( Knx^n + K_{n-1}x^{n-1} + \cdots + K_1x + K_0 )</td>
<td>( K_n, \ldots, K_0 )</td>
</tr>
</tbody>
</table>

There is a similar table for the case in which the complementary equation is of Euler-Cauchy type, obtained simply by the change of variables \( x \to \log x \) from the table above.

<table>
<thead>
<tr>
<th>( r(x) ) with Euler-Cauchy</th>
<th>Trial Function</th>
<th>Solve for</th>
</tr>
</thead>
<tbody>
<tr>
<td>( kx^n ) (( n \in \mathbb{Z}, n \geq 0 ))</td>
<td>( Kn\log^n x + \cdots + K_1 \log x + K_0 )</td>
<td>( K_n, \ldots, K_0 )</td>
</tr>
</tbody>
</table>

• If the trial function listed above is already a solution of the complementary equation, multiply the trial function by \( x \) in the constant-coefficient case (\( \log x \) in the Euler-Cauchy case). Repeat if necessary.

• If \( r(x) \) is a sum or product of the types of functions listed above, write the trial function as a corresponding sum or product.

This method does not always give a solution, but any incorrect trial function can be detected, because there will be no constant solution for its coefficient(s).

6.2 Variation of parameters

There is a more general method. If we have found linearly independent solutions \( y_1 \) and \( y_2 \) to the complementary equation, then write

\[ y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \] (8)

There will be a great deal of freedom left in finding functions \( u_1, u_2 \) so that this equation is a valid solution, and we will make use of this freedom.

• From eq. (8), derive expressions for \( y'_p(x) \) and \( y''_p(x) \) and substitute them into the differential equation. When deriving \( y'_p(x) \), set

\[ u'_1(x)y_1(x) + u'_2(x)y_2(x) = 0. \] (9)

This is an arbitrary constraint, but it still allows a general solution, and it simplifies the expression for \( y''_p(x) \) when the second derivative is taken.

• Because \( y_1 \) and \( y_2 \) solve the complementary equation, the terms proportional to \( u_1 \) and \( u_2 \) drop out of the differential equation, leaving the following equation:

\[ u'_1(x)y'_1(x) + u'_2(x)y'_2(x) = r(x). \] (10)

• Equations (9) and (10) form a linear system of two equations in the two unknowns \( u'_1 \) and \( u'_2 \). Solve the system, and then integrate to obtain \( u_1 \) and \( u_2 \).

• Substitute these solutions into eq. (8), and then write the general solution as in eq. (5).

The determinant of the linear system of equations happens to be the Wronskian of \( y_1 \) and \( y_2 \).

Although variation of parameters is more general, the method of undetermined coefficients is preferable when it works, because differentiation is much easier than integration.
7 Series solutions

There are many important differential equations whose solutions are new “special” functions bearing the same name as the equations, such as Airy, Legendre, Bessel, hypergeometric. We often approach these special functions through power series.

7.1 Power series

- A power series about the point \( x = x_0 \) is a series of the form

  \[
  f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m
  \]

  \[
  = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots
  \]

- A power series has a radius of convergence \( R \), such that the series converges for \( |x - x_0| < R \) and diverges for \( |x - x_0| > R \). The values \( x \) for which \( |x - x_0| < R \) are said to lie within the radius of convergence.

- Moreover, power series converge absolutely within their radius of convergence. Therefore, where power series converge, they can be added, multiplied, differentiated, or integrated term by term.

- One way to determine the radius of convergence \( R \) is through the formula

  \[
  R^{-1} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| .
  \]

- \( f(x) \) is analytic at \( x = x_0 \) if it has a power series expansion about this point with a positive radius of convergence. Equivalently, \( f(x) \) is analytic at \( x = x_0 \) if it is infinitely differentiable at this point.

- Convergent power series about \( x_0 \) form a vector space, for which the infinite set

  \[
  \{ 1, (x - x_0), (x - x_0)^2, (x - x_0)^3, \ldots \}
  \]

  is a basis.

- A polynomial is a special case of a power series. For any nonzero polynomial, there exists a nonnegative integer \( k \), which is the degree of the polynomial, such that \( a_k \neq 0 \), but \( a_m = 0 \) whenever \( m > k \). Every polynomial converges for all values of \( x \) and is thus analytic with an infinite radius of convergence.

The idea of series solutions is to expand the ODE and its general solution in the power series basis, and identify the solution by finding the coefficients \( a_m \) of \((x - x_0)^m\).

7.2 Ordinary and singular points of ODE

Consider a second-order linear ODE, written in standard form:

\[
y'' + p(x)y' + q(x)y = r(x).
\]  \hspace{1cm} (11)

- If \( p, q, \) and \( r \) are analytic at \( x = x_0 \), then there exist series solutions to the ODE, and \( x = x_0 \) is called an ordinary point of the differential equation. Otherwise, \( x = x_0 \) is a singular point.

- In the special case that \( r(x) = 0 \): if \( x = x_0 \) is a singular point, but \((x - x_0)p(x)\) and \((x - x_0)^2q(x)\) are analytic functions at \( x = x_0 \), then \( x = x_0 \) is a regular singular point of the differential equation. Otherwise, \( x = x_0 \) is an essential singularity.

Standard form is important for the analysis of singularities. However, to actually produce series solutions, we often work with equations in a non-standard form where any denominators have been multiplied through the equation, so that the coefficients of \( y'', y', y \) and the non-homogeneous term are polynomials or power series in \( x \).
7.3 Series solutions about an ordinary point

If $x = x_0$ is an ordinary point of eq. (11), then write

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

and substitute this series into the ODE.

- Expand $p$, $q$, and $r$ as power series around $x = x_0$, using the Taylor expansion if necessary.
- Group terms by powers of $(x - x_0)$. It is typically necessary to shift the summation index in some of the terms so that the different series can be combined.
- Set coefficients of each basis element $(x - x_0)^m$ equal on both sides of the equation. Pay special attention to small values of $m$ which might have to be treated separately, case by case.
- Solve for all of the coefficients $a_m$ recursively. Start with the lowest power of $(x - x_0)$, and work your way upwards until you arrive at a general recursion relation for sufficiently large $m$.
- The general solution will contain two undetermined constants among the $a_m$, from which all of the other coefficients can be derived. In fact, the first two coefficients can be interpreted as the initial values $a_0 = y(x_0)$ and $a_1 = y'(x_0)$.
- The radius of convergence of the series solution extends to the nearest singular point.

7.4 Series solutions about a regular singular point: Frobenius method

We assume that the differential equation itself is real. That is, it is defined for real values of $x$, and all functions appearing in the ODE are real-valued.

**Fuchs’s Theorem** If $x = x_0$ is a regular singular point of eq. (11), then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{m=0}^{\infty} a_m (x - x_0)^m,$$

where $r \in \mathbb{C}$ and $a_0 \neq 0$. (12)

**Frobenius Method**

- Substitute eq. (12) into the ODE. Expand $p$ and $q$ as power series around $x = x_0$, so that the entire equation is written in the form of a series expansion. The lowest power of $(x - x_0)$ appearing in the equation will be $(x - x_0)^r$, and all terms of this order are proportional to $a_0$.
- Take the entire coefficient of $(x - x_0)^r$ in the equation, and divide the equation through by $a_0$. The result is the **indicial equation**, which is a quadratic equation in $r$.
- By Fuchs’s theorem, one of the roots of the indicial equation yields a series solution. We can think of it as the larger root:
  - If the two roots are real and distinct, take the larger root.
  - If there is a double (real) root, its value yields a solution.
  - If the two roots are complex, neither is larger, and in fact they yield two independent solutions, giving the complete general solution to the ODE.
- Take the root identified above, substitute it in eq. (12), and obtain the coefficients $a_m$ recursively as in the previous power series method. The coefficient $a_0$ will be the single undetermined constant.
Fuchs’s Theorem, continued  It is possible to find the second linearly independent solution using the series method, as follows. Let \( r_1 \) and \( r_2 \) denote the two roots of the indicial equation (where possibly \( r_1 = r_2 \)).

- If \( r_1 = r_2 \), let \( r = r_1 = r_2 \) be the value of this double root. Then the two independent solutions take the form

\[
y_1 = (x - x_0)^r \sum_{m=0}^{\infty} a_m (x - x_0)^m \\
y_2 = \log(x - x_0) y_1 + (x - x_0)^r \sum_{m=0}^{\infty} A_m (x - x_0)^m
\]

- If \( r_1 \neq r_2 \) but \( r_1 - r_2 \) is an integer, then let \( r_1 \) be the larger of the two roots. Then

\[
y_1 = (x - x_0)^{r_1} \sum_{m=0}^{\infty} a_m (x - x_0)^m \\
y_2 = k \log(x - x_0) y_1 + (x - x_0)^{r_2} \sum_{m=0}^{\infty} A_m (x - x_0)^m,
\]

where possibly \( k = 0 \).

- If \( r_1 \neq r_2 \) and \( r_1 - r_2 \) is not an integer, then \( r_1 \) and \( r_2 \) give independent series solutions. This case includes all complex roots.

\[
y_1 = (x - x_0)^{r_1} \sum_{m=0}^{\infty} a_m (x - x_0)^m \\
y_2 = (x - x_0)^{r_2} \sum_{m=0}^{\infty} A_m (x - x_0)^m
\]

In all of these cases, the series expressions are substituted into the ODE, and the coefficients \( k, a_m \) and \( A_m \) are found recursively.