Consider the following differential equation.

\[(4 - x^2)y'' + 6y = 0.\]

1. [1 point] What are the singular points of the differential equation? Are they regular singular points, or essential singularities?

2. [8 points] Find the general series solution about the point \(x_0 = 0\).

3. [2 points] Find a nonzero polynomial solution.

4. [1 point] Given initial values \(y(0) = 1, y'(0) = 0\), what are the first three nonvanishing terms in the series solution?

Solution:

1. In standard form, the equation would be written as \(y'' + \frac{6}{4 - x^2}y = 0\), which is \(y'' + py' + qy = 0\) with \(p = 0\) and \(q = \frac{6}{4 - x^2}\). The coefficient of \(y\) is analytic everywhere except at \(x = \pm 2\), so these are the only singular points.

   To classify the singularity at \(x_0 = 2\), we compute \((x - x_0)p = 0\) and \((x - x_0)^2q = 6(2 - x)/(2 + x)\). Both of these functions are analytic at the point \(x_0 = 2\), so this is a regular singular point.

   For \(x_0 = -2\), we have \((x - x_0)p = 0\) and \((x - x_0)^2q = 6(x + 2)/(x - 2)\), which are both analytic at \(x_0 = -2\), so this is also a regular singular point.

2. Note that \(x = 0\) is an ordinary point, not a singular point. So there exists a power series solution around this point. Write the series expansion for \(y\) and its first two derivatives:

   \[
y = \sum_{m=0}^{\infty} a_m x^m \quad (1)
   \]

   \[
y' = \sum_{m=1}^{\infty} ma_m x^{m-1} \quad (2)
   \]

   \[
y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} \quad (3)
   \]

   The terms appearing in the differential equation then have the following series expansions:

   \[
   6y = \sum_{m=0}^{\infty} 6a_m x^m
   \]

   \[
   -x^2 y'' = \sum_{m=0}^{\infty} -m(m - 1)a_m x^m
   \]

   \[
   4y'' = \sum_{m=0}^{\infty} 4m(m - 1)a_m x^{m-2} = \sum_{m=2}^{\infty} 4m(m - 1)a_m x^{m-2}
   \]

   where the second equality in the last line recasts this term as a proper power series with nonnegative powers, which is valid because the \(m = 0\) and \(m = 1\) terms are seen to be zero.
We shift \( m \to m+2 \) in the summation index of the last equation, in order to get the powers of \( x \) to match the others:

\[
4y'' = \sum_{m=0}^{\infty} 4(m+2)(m+1)a_{m+2}x^m
\]

Now we substitute each term back into the differential equation:

\[
0 = \sum_{m=0}^{\infty} [4(m+2)(m+1)a_{m+2} - m(m-1)a_m + 6a_m]x^m
\]

Then we equate the coefficients of each power of \( x \). For all \( m \), we have

\[
0 = 4(m+2)(m+1)a_{m+2} + (-m^2 + m + 6)a_m
\]

leading to the recursive relation for series coefficients,\(^1\)

\[
a_{m+2} = \frac{m^2 - m - 6}{4(m+2)(m+1)}a_m \quad \text{for } m \geq 0,
\]

which can be simplified to

\[
a_{m+2} = \frac{m-3}{4(m+1)}a_m \quad \text{for } m \geq 0.
\]

The full series solution can be presented as

\[
\begin{align*}
y &= \sum_{m=0}^{\infty} a_m x^m, \\
a_0 \text{ and } a_1 \text{ are undetermined,} \\
a_{m+2} &= \frac{m-3}{4(m+1)}a_m \quad \text{for } m \geq 0.
\end{align*}
\]

3. There is a polynomial solution if the recursion relation can terminate, i.e. if there is some value of \( j \) for which \( a_m = 0 \) whenever \( m \geq j \). From the recursive relation above, we see that we can have \( a_{m+2} = 0 \) if \( m - 3 = 0 \), which has a nonnegative solution at \( m = 3 \). Thereafter, all odd series coefficients will vanish. In the case where \( a_0 = 0 \) so that all even series coefficients vanish, we have a polynomial solution of degree 3, which is

\[
y = a_1 \left( x - \frac{x^3}{4} \right).
\]

4. With initial values \( y(0) = 1 \) and \( y'(0) = 0 \), it follows from equations (1) and (2) that \( a_0 = 1 \) and \( a_1 = 0 \), so then the particular series solution is

\[
\begin{align*}
y &= \sum_{m=0}^{\infty} a_m x^m, \\
a_0 &= 1, \\
a_1 &= 0, \\
a_{m+2} &= \frac{m-3}{4(m+1)}a_m \quad \text{for } m \geq 1.
\end{align*}
\]

In this case, all the odd terms vanish. The recursive relation gives

\[
\begin{align*}
a_2 &= -\frac{3a_0}{4} = -\frac{3}{4}, \\
a_3 &= 0, \\
a_4 &= -\frac{a_2}{12} = \frac{1}{16}.
\end{align*}
\]

The series begins

\[
y = 1 - \frac{3x^2}{4} + \frac{x^4}{16} + \cdots
\]

\(^1\)At this point, we should check that we have not divided by zero. In this case, there is no problem, because \( 4(m+1)(m+2) \neq 0 \) for all \( m \geq 0 \). Otherwise, we must use eq. (4) as written.