1. Verify that $y_1 = e^x$ is a solution to the differential equation
\[ xy'' + (5 - 2x)y' + (x - 5)y = 0. \] (1)

**Solution:** $y_1'' = y_1' = y_1 = e^x$, so $xy''_1 + (5 - 2x)y_1' + (x - 5)y_1 = [x + (5 - 2x) + (x - 5)]e^x = 0.$

2. Use the method of reduction of order to find the general solution to equation (1).

**Solution:** Write $y_2 = uy_1$, and use the differential equation to solve for the function $u$:

\[
\begin{align*}
y_2 & = e^x u \\
y_2' & = e^x u' + e^x u \\
y_2'' & = e^x u'' + 2e^x u' + e^x u
\end{align*}
\]

\[
0 = xy''_2 + (5 - 2x)y'_2 + (x - 5)y_2 = e^x [xu'' + 5u']
\]

\[
u'' \frac{u'}{u} = \frac{-5}{x}
\]

\[
\log u' = -5 \log x
\]

\[
u' = x^{-5}
\]

\[
u = -\frac{1}{4}x^{-4}
\]

where we have put in arbitrary integration constants because we only need a single particular solution that is linearly independent of $y_1$ (and linear independence is guaranteed by the method). Therefore the solution $y_2$ is

\[ y_2 = uy_1 = -\frac{e^x}{4x^4} \]

and the general solution to the differential equation can be written as

\[ y = c_1 e^x - c_2 \frac{e^x}{4x^4} \]

(It is possible to absorb the factor $-\frac{1}{4}$ of the second term into the arbitrary constant $c_2$.)

3. Find the general solution to $9\ddot{f}(t) - 24\dot{f}(t) + 16f(t) = 0$, and the particular solution with $f(0) = 4$, $\dot{f}(0) = 0$.

**Solution:**

Use a trial solution $f = e^{\lambda t}$. The characteristic equation is $9\lambda^2 - 24\lambda + 16 = 0$, and there is a double root at $\lambda = 4/3$. So the general solution to the ODE, in terms of real-valued functions, is

\[ f(t) = (c_1 + tc_2)e^{4t/3}. \]
We need to take its derivative to substitute the initial values:

\[
\dot{f}(t) = e^{4t/3} \left( \frac{4}{3} c_1 + c_2 + \frac{4}{3} c_2 t \right).
\]

The initial values impose the following two equations for the constants:

\[
\begin{align*}
4 &= c_1 \\
0 &= \frac{4}{3} c_1 + c_2
\end{align*}
\]

The solutions for the constants are \(c_1 = 4, c_2 = -16/3\), resulting in the particular solution:

\[
f(t) = \left( 4 - \frac{16}{3} t \right) e^{4t/3}.
\]

4. Find the general solution to \(t^2 \ddot{r}(t) - 3t \dot{r}(t) + 5r(t) = 0\), and the particular solution with \(r(1) = 1, \dot{r}(1) = -3\).

**Solution:**

Use a trial solution \(r = t^m\). The characteristic equation is \(m^2 - 4m + 5 = 0\), with roots at \(m = 2 \pm i\).

So the general solution to the ODE, in terms of real-valued functions, is

\[
r(t) = t^2 (c_1 \log t + c_2 \cos \log t).
\]

We need to take its derivative to substitute the initial values:

\[
\dot{r}(t) = t^2 ([c_1 + 2c_2] \cos \log t + [2c_1 - c_2] \sin \log t).
\]

The initial values impose the following two equations for the constants:

\[
\begin{align*}
1 &= c_2 \\
-3 &= c_1 + 2c_2
\end{align*}
\]

The solutions for the constants are \(c_1 = -5, c_2 = 1\), resulting in the particular solution

\[
r(t) = t^2 (\log t - 5 \sin \log t).
\]

**Solution of a different problem:**

This tutorial sheet was initially distributed with a typo in the problem, which was given as:

Find the general solution to \(t^2 \ddot{r}(t) - 4t \dot{r}(t) + 5r(t) = 0\), and the particular solution with \(r(1) = 1, \dot{r}(1) = -3\).

In this case, the characteristic equation would be \(m^2 - 5m + 5 = 0\), with roots at \(m = (5 \pm \sqrt{5})/2\).

The general solution is

\[
r(t) = c_1 t^{(5+\sqrt{5})/2} + c_2 t^{(5-\sqrt{5})/2}
\]

whose derivative is

\[
\dot{r}(t) = c_1 \frac{5 + \sqrt{5}}{2} t^{(5+\sqrt{5})/2} + c_2 \frac{5 - \sqrt{5}}{2} t^{(5-\sqrt{5})/2}
\]

The initial values impose the following two equations for the constants:

\[
\begin{align*}
1 &= c_1 + c_2 \\
-3 &= c_1 \frac{5 + \sqrt{5}}{2} + c_2 \frac{5 - \sqrt{5}}{2}
\end{align*}
\]

giving \(c_{1,2} = (5 \mp 11\sqrt{5})/10\) and the solution

\[
r(t) = \frac{5 - 11\sqrt{5}}{10} t^{(5+\sqrt{5})/2} + \frac{5 + 11\sqrt{5}}{10} t^{(5-\sqrt{5})/2}
\]