1 Kronecker delta

The Kronecker delta function is defined by
\[ \delta_{ab} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases} \]

Typically, the indices \( a \) and \( b \) take integer values.

Here is one example of an application. For an indexed set \( \{a_i, i \in S\} \),
\[ \sum_{i \in S} a_i \delta_{ik} = a_k, \]
assuming that \( k \in S \). If \( k \notin S \), then the series is 0. We often see applications where \( S \) is the set of nonnegative integers, or all integers.

2 Complex numbers

The space of complex numbers is denoted by \( \mathbb{C} \). Any complex number can be written in the form \( a + ib \), where \( a \) and \( b \) are real numbers \( (a, b \in \mathbb{R}) \) and \( i^2 = -1 \) by definition.

If \( z = a + ib \) is a complex number, with \( a, b \in \mathbb{R} \), then its complex conjugate is \( z^* = a - ib \). Its magnitude is \( |z| = \sqrt{zz^*} = \sqrt{a^2 + b^2} \), which is a nonnegative real number. The magnitude of any nonzero complex number is strictly positive. The complex conjugate is sometimes denoted by \( \bar{z} \).

2.1 Quadratic formula

The quadratic equation \( ax^2 + bx + c = 0 \) has two solutions, also called roots,
\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \]
The two roots coincide if the discriminant \( b^2 - 4ac \) is 0.
Suppose that \( a, b, \) and \( c \) are real. We identify three cases:

- If \( b^2 - 4ac > 0 \), then there are two distinct real roots.
- If \( b^2 - 4ac = 0 \), then there is a real-valued double root.
- If \( b^2 - 4ac < 0 \), then the roots are distinct and complex, and they are complex conjugates of each other.

2.2 Roots of polynomials

In general, a polynomial of degree \( n \) has \( n \) roots, which are complex numbers that may not all be distinct. This statement can be expressed in the equation
\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - x_1)(x - x_2) \cdots (x - x_n). \]
Here, \( x_1, \ldots, x_n \) are the roots. The statement is true regardless of whether the coefficients \( a_i \) are real or complex.
If any of the roots are known, the problem of finding the rest of the roots can be reduced to analyzing a polynomial of lower degree. Suppose that \( p_n(x) \) is a given polynomial of degree \( n \), and \( x_n \) is a number satisfying \( p_n(x_n) = 0 \). Then
\[
p_n(x) = (x - x_n) \left( b_{n-1}x^{n-1} + \cdots + b_1x + b_0 \right),
\]
and the coefficients \( b_i \) can be found by multiplying out the right-hand side and equating like powers of \( x \).

### 2.3 Roots of unity
The equation \( x^n = 1 \) has \( n \) complex roots, which are \( x_k = e^{\frac{2\pi ik}{n}} \) for integer values of \( k \).

Notice that it is sufficient to take \( 0 \leq k \leq n - 1 \), or \( 1 \leq k \leq n \), since any other choice of index produces a value equivalent to one of these.

### 3 Euler’s formula
For any real number \( x \),
\[
e^{ix} = \cos x + i\sin x.
\]
It follows that we can write the cosine and sine functions as linear combinations of exponentials:
\[
\cos x = \frac{e^{ix} + e^{-ix}}{2}, \\
\sin x = \frac{e^{ix} - e^{-ix}}{2i}.
\]
It is then possible to derive all of the basic trigonometric identities as consequences of exponential multiplication (\( e^x e^y = e^{x+y} \)) and differentiation (\( \frac{d}{dx} e^x = e^x \)).

### 4 Vector spaces
- A **real vector space** \( V \) is a set of objects called vectors which is closed under two operations:
  - Addition. If \( v, w \in V \), then \( v + w \in V \). There is an identity element denoted by \( 0 \). Every vector has an additive inverse. Addition is commutative and associative.
  - Scalar multiplication. If \( v \in V \) and \( a \in \mathbb{R} \), then \( av \in V \). There is an identity element in \( \mathbb{R} \) denoted by \( 1 \).

The two operations are linked by the distributive property.

- A **complex vector space** is defined similarly, with \( \mathbb{R} \) above replaced by \( \mathbb{C} \).

- A set of vectors \( \{v_i\} \) is **linearly independent** if the only solution to the equation \( \sum_i a_i v_i = 0 \) is that \( a_i = 0 \) for every \( i \).

- If there exists a set \( B = \{v_i\} \) of linearly independent vectors such that every vector \( v \in V \) has a decomposition \( v = \sum_i a_i v_i \), then \( B \) is a **basis** of \( V \). The size of the set, \( |B| \), is the **dimension** of \( V \).

Linear independence of the basis guarantees uniqueness of the decomposition.

- An **inner product** on a **real vector space** \( V \) is a map \( (\ ,\ ) : V \times V \to \mathbb{R} \) satisfying the following properties.
  1. Symmetry: \( \langle u|v \rangle = \langle v|u \rangle \) for any \( u, v \in V \)
  2. Bilinearity: \( \langle a_1 u + a_2 v|w \rangle = a_1 \langle u|w \rangle + a_2 \langle v|w \rangle \) and \( \langle u|a_1 v + a_2 w \rangle = a_1 \langle u|v \rangle + a_2 \langle u|w \rangle \) for any \( u, v, w \in V \) and \( a_1, a_2 \in \mathbb{R} \).
3. Positive definiteness: for any \( u \in V \), \( \langle u|u \rangle \geq 0 \),
and \( \langle u|u \rangle = 0 \iff u = 0 \).

- An inner product on a complex vector space \( V \) is a map \( \langle \cdot | \cdot \rangle : V \times V \rightarrow \mathbb{C} \) satisfying the following properties.
  1. Conjugate symmetry: \( \langle u|v \rangle = \langle v|u \rangle^* \) for any \( u, v \in V \)
  2. Complex bilinearity: \( \langle a_1 u + a_2 v|w \rangle = a_1^* \langle u|w \rangle + a_2^* \langle v|w \rangle \)
and \( \langle u|a_1 v + a_2 w \rangle = a_1 \langle u|v \rangle + a_2 \langle u|w \rangle \)
for any \( u, v, w \in V \) and \( a_1, a_2 \in \mathbb{R} \).
  3. Positive definiteness: for any \( u \in V \), \( \langle u|u \rangle \) is real, \( \langle u|u \rangle \geq 0 \),
and \( \langle u|u \rangle = 0 \iff u = 0 \).

- A vector space together with a specific choice of inner product is called an inner product space.
The following points regarding orthogonality and projection pertain to inner product spaces.
- A pair of vectors \( u, v \) is orthogonal if \( \langle u|v \rangle = 0 \). A set of vectors \( S = \{v_i\} \) is orthogonal if \( \langle v_i|v_j \rangle = 0 \) whenever \( i \neq j \).

- Orthogonal projection. Suppose that \( S = \{w_i\} \) is a set of orthogonal vectors. If a vector \( v \) is linearly dependent on \( S \), meaning that there exist real-valued coefficients \( a_i \) such that \( v = \sum_i a_i w_i \),
then these coefficients can be obtained from the following relation:

\[
a_i = \frac{\langle v|w_i \rangle}{\langle w_i|w_i \rangle}
\]
The relation is derived by taking the inner product of the expansion equation with the basis element \( w_i \). If \( S \) is a basis for the vector space \( V \), then every vector \( v \in V \) has such an expansion.

5 Systems of linear equations

Systems of two linear equations in two variables take the form

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{align*}
\]

where the \( a_{ij} \) and \( b_i \) are constants.

You can probably think of several ways to solve this system for \( x_1 \) and \( x_2 \).

One way is to focus first on \( x_1 \). Eliminate \( x_2 \) by multiplying eq. (1) by \( a_{22} \) and eq. (2) by \( -a_{12} \)
and then summing the results. Then the left-hand side is proportional to \( x_1 \). Divide through by its prefactor to get the value of \( x_1 \). Then, repeat a similar process to solve for \( x_2 \), starting with multiplying eq. (1) by \( a_{21} \) and eq. (2) by \( -a_{11} \).

In this module we sometimes encounter larger systems of linear equations, but they will be readily approachable through elimination of one variable at a time. For example, for the triangular system

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
& \vdots \\
a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n
\end{align*}
\]
solve the last equation for \( x_n \), then insert this value into the previous equation to solve for \( x_{n-1} \), and so on.
6 Series

- In this module, we work with two kinds of infinite series, namely
  \[ \sum_{m=-\infty}^{\infty} a_m \]
  summed over all integer values of the index \( m \), or
  \[ \sum_{m=0}^{\infty} a_m \]
  summed over nonnegative integer values of \( m \). The second type can be used for solving differential equations, as well as for Fourier series.

- A series that has a limiting value is said to converge, and it can be manipulated freely with algebraic operations if it converges absolutely, i.e. if the series \( \sum_{m} |a_m| \) converges.

- Infinite geometric series:
  \[ \sum_{m=0}^{\infty} q^m = \frac{1}{1-q} \quad \text{if} \quad |q| < 1, \quad q \in \mathbb{C}. \]
  If \(|q| \geq 1\), the series does not converge.

- Finite geometric series:
  \[ \sum_{m=0}^{r} q^m = \frac{1 - q^{r+1}}{1-q} \quad \text{if} \quad q \neq 1, \quad q \in \mathbb{C}. \]
  If \( q = 1 \), the series obviously sums to \( r + 1 \).

- The following are Taylor series expansions around \( x = 0 \).
  \[ \log(1 + x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m \]
  \[ e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} \]
  Either from Euler's identity applied to the last line, or from direct differentiation, you can derive the power series for \( \sin x \) and \( \cos x \).

7 Calculus

The following concepts should be thoroughly understood and applied from memory wherever appropriate.

- Product rule
  \[ \frac{d}{dx} (f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx} \]

- Chain rule
  \[ \frac{d}{dx} (f(g(x))) = \frac{df}{dg} \cdot \frac{dg}{dx} \]

- Integration by parts
  \[ \int u \, dv = uv - \int v \, du \]
  Be careful in applying integration by parts: it should only be done piecewise on intervals on which the functions are continuous.

- Method of partial fractions for integrating rational functions