# Fourier Analysis

**Contents**

1. Periodic functions ........................................... 1
2. Fourier Series ........................................... 1
   2.1 Even and odd functions; half-range expansion ............... 2
   2.2 Complex Fourier Series .................................... 3
   2.3 Parseval’s Theorem .................................... 3
3. Fourier Transform ........................................... 3
   3.1 Some properties of the Fourier transform ............... 4
   3.2 Dirac delta ........................................ 4
   3.3 Plancharel’s theorem .................................... 4
   3.4 Convolution [omitted in 2017] .................................... 5
   3.5 Gaussian distribution .................................... 5
4. Discrete Fourier Transform .................................... 5
   4.1 Parseval’s Theorem for DFT ......................... 6

## 1. Periodic functions

- A function $f(t)$ is **periodic** in $t$ if there exists some nonzero number $a$, called a **period**, such that $f(t) = f(t + a)$ for all values of $t$. The period may also refer to an interval of that length on which the function is defined.

- The **smallest positive** period of a given function is called its **fundamental period**.

## 2. Fourier Series

The real Fourier series expansion of a function $f(t)$ of fundamental period $L$ takes the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{L} \right).$$

We refer to $a_0/2$ and all the $a_n$ and $b_n$ as **Fourier coefficients**, and to the constant, cosine, and sine functions appearing here as **Fourier modes**.

Different sources use different conventions for the general presentation of the series and the definitions of the coefficients. However, the conventions work out so that the real Fourier series expansion of any specific function is unique.

- Not every periodic function admits a convergent Fourier series expansion. The following **Dirichlet conditions** are a list of criteria that are collectively sufficient for the existence and convergence of the Fourier series expansion of a periodic function:
  - finitely many discontinuities within one period
  - finitely many maxima and minima within one period
  - discontinuities are finite
Convergence means that truncations of the Fourier series give increasingly close approximations to the function \( f(t) \). In other words, taking just the terms indexed by 0, 1, ..., \( k \) for some nonnegative integer \( k \) approximates the function as the value of \( k \) is increased towards infinity.

At a discontinuity point, the Fourier series expansion of a function will converge to the average of the left- and right-hand limits.

Real functions of period \( L \) may be viewed as an infinite-dimensional vector space with the following inner product:

\[
\langle f | g \rangle = \int_{t_0}^{t_0+L} f(t)g(t) \, dt.
\]

In this inner product space, the constant function and the cosines and sines appearing in equation (1) form an orthogonal set. They form a basis of periodic functions satisfying the Dirichlet conditions. Thus the Fourier series expansion is justified.

Because the functions are periodic, \( t_0 \) can be chosen arbitrarily without affecting the value of \( \langle f | g \rangle \).

It follows from orthogonal projection that the Fourier coefficients are given by the following Euler formulas:

\[
a_0 = \frac{2}{L} \int_{t_0}^{t_0+L} f(t) \, dt
\]
\[
a_n = \frac{2}{L} \int_{t_0}^{t_0+L} f(t) \cos \left( \frac{2\pi nt}{L} \right) \, dt
\]
\[
b_n = \frac{2}{L} \int_{t_0}^{t_0+L} f(t) \sin \left( \frac{2\pi nt}{L} \right) \, dt
\]

It is very important to note that these expressions pertain specifically to the coefficients as defined in (1). Different choices of conventions may result in formulas that differ by a constant factor.

Although the choice of \( t_0 \) is arbitrary, a judicious choice will simplify calculations in practice.

2.1 Even and odd functions; half-range expansion

Definitions:

- \( f(x) \) is **even** if \( f(-x) = f(x) \);
- \( f(x) \) is **odd** if \( f(-x) = -f(x) \).

Fourier series for even functions will have only cosine and constant terms (note that \( \cos(0)=1 \)).

Fourier series for odd functions will have only sine terms.

It is sometimes useful to write Fourier series of a function \( f(t) \) defined on a finite interval \([0, L]\). This can be done by extending the function to the entire real axis in such a way that it becomes periodic. There are three natural ways to do so.

- **Simple periodic extension:**
  \[ f_s(t) = f(t) \text{ if } 0 \leq t < L, \quad \text{and } f_s(t+L) = f_s(t) \text{ for all } t. \]

- **Even extension:**
  \[ f_e(t) = \begin{cases} f(t) & \text{if } 0 \leq t < L, \\ f(-t) & \text{if } -L \leq t < 0, \end{cases} \quad \text{and } f_e(t+2L) = f_e(t) \text{ for all } t. \]

- **Odd extension:**
  \[ f_o(t) = \begin{cases} f(t) & \text{if } 0 \leq t < L, \\ -f(-t) & \text{if } -L \leq t < 0, \end{cases} \quad \text{and } f_o(t+2L) = f_o(t) \text{ for all } t. \]

Fourier series of even and odd extensions are called half-range expansions, because the initial interval \([0, L]\) has half the length of the fundamental period of the extension function.
2.2 Complex Fourier Series

The complex Fourier series expansion of a function \( f(t) \) of period \( L \) can be written as

\[
f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi in}{L}t},
\]

where the coefficients are given by

\[
c_n = \frac{1}{L} \int_{t_0}^{t_0+L} f(t) e^{-\frac{2\pi in}{L}t} \, dt.
\]

All properties of the complex Fourier series are analogous to, and derivable from, the real Fourier series, via Euler’s formula.

If \( f(t) \) takes real values, then the complex Fourier coefficients will be related among themselves by \( c_{-n} = c_n^\ast \).

Euler’s formula effectively gives a change of basis of the vector space of periodic functions satisfying the Dirichlet conditions. The pair of functions \( \{\cos \left( \frac{2\pi nt}{L} \right), \sin \left( \frac{2\pi nt}{L} \right)\} \) can be exchanged for the pair \( \{e^{\frac{2\pi in}{L}t}, e^{-\frac{2\pi in}{L}t}\} \), by a unitary linear transformation, which preserves the property of their orthogonality.

The orthogonality of Fourier modes is expressed in the following relation.

\[
\frac{1}{L} \int_{t_0}^{t_0+L} (e^{\frac{2\pi im}{L}t})^\ast e^{\frac{2\pi in}{L}t} = \delta_{mn}.
\]

2.3 Parseval’s Theorem

For a function of period \( L \) whose real Fourier series expansion is written in the form above, the following equation is true:

\[
\frac{1}{L} \int_{t_0}^{t_0+L} f(t)^2 \, dt = \left( \frac{a_0}{2} \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).
\]

For a complex Fourier series, we have instead

\[
\frac{1}{L} \int_{t_0}^{t_0+L} f(t)^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2.
\]

This theorem is a straightforward consequence of the orthogonality of Fourier modes.

3 Fourier Transform

The Fourier transform is defined by \( \mathcal{F}: f(t) \to \hat{f}(\omega) \), where

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt.
\]

Inverse Fourier transform:

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega.
\]

This pair of transforms can be interpreted as a continuous analog of a Fourier series: the period \( L \) is taken to infinity, and all values of frequencies are permitted. The values of the Fourier transform are playing the roles of the Fourier coefficients, and \( \omega \) plays the role of the index \( n \). The Inverse Fourier Transform equation is then the analog of the Fourier series expansion.

Different sources use different conventions for the definition of the Fourier transform: the formulas may be complex-conjugated compared to the ones above, and factors of \( \sqrt{2\pi} \) may be shifted. What is important is that the inverse Fourier transform truly inverts the operation of the Fourier transform.
3.1 Some properties of the Fourier transform

With the definition given above, we can prove the following properties. The statements below are true if both sides of the equation are well defined.

- **Linearity**: if \( a \) and \( b \) are any constants, then \( \mathcal{F}[af(t) + bg(t)] = a\mathcal{F}[f(t)] + b\mathcal{F}[g(t)] \).
- **Scaling**: if \( a > 0 \), then \( \mathcal{F}[f(at)] = \frac{1}{a} \tilde{f} \left( \frac{\omega}{a} \right) \).
- **Translation**: if \( a \in \mathbb{R} \), then \( \mathcal{F}[f(t + a)] = e^{i\omega a} \tilde{f}(\omega) \).
- **Exponential multiplication**: if \( \alpha \in \mathbb{C} \), then \( \mathcal{F}[e^{\alpha t}f(t)] = \tilde{f}(\omega + i\alpha) \).
- **Differentiation**: \( \mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega) \).
- **Symmetry or duality**: \( f(-\omega) \) is the Fourier transform of \( \tilde{f}(t) \).

3.2 Dirac delta

The Dirac delta is not inherently associated to Fourier analysis, but this is the context in which we are first meeting it.

The Dirac delta “function” is better described as a generalized function or distribution. This is because its description in terms of functional values obscures the essential property of \( \delta(t) \), which is that \( \delta(t) = 0 \) at any point \( t \neq 0 \), and

\[
\int_a^b \delta(t) \, dt = \begin{cases} 1 & \text{if } a < 0 < b, \\ 0 & \text{otherwise.} \end{cases}
\]  

(5)

The Dirac delta should only be used inside integrals.

One way of defining the Dirac delta distribution is as the limit of any spiky shape of unit area, as the width goes to zero and the height goes to infinity, such that the area remains constant. For example, \( \delta(t) = \lim_{k \to 0} f_k(t) \), where

\[
f_k(t) = \begin{cases} \frac{1}{k} & \text{for } 0 < t < k, \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore the value of \( \delta(t) \) is 0 whenever \( t \neq 0 \).

One very useful relation is

\[
\int_{-\infty}^{\infty} \delta(t - a) g(t) \, dt = g(a)
\]

for any function \( g(t) \) that is continuous at the point \( t = a \). Any integral involving the Dirac delta can be evaluated using the equations above and basic integral relations, notably change of variables inside the argument of the delta function. With a general functional argument of the delta function, we can replace it with simpler ones as follows:

\[
\delta(h(t)) = \sum_i \frac{\delta(t - t_i)}{h'(t_i)}
\]

where the sum runs over the solutions to the equation \( h(t) = 0 \), which are indexed by \( i \).

In analogy to eq. (2), we can now use the Dirac delta to express the orthogonality of Fourier transform modes:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{-i\omega_1 t})^* e^{-i\omega_2 t} \, dt = \delta(\omega_1 - \omega_2) .
\]

(6)

3.3 Plancharel’s theorem

If \( \tilde{f}(\omega) \) is the Fourier transform of \( f(t) \), as defined in eq. (3), then

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| \tilde{f}(\omega) \right|^2 \, d\omega ,
\]

provided that both of these integrals are well defined and converge.

This is the continuous analog of Parseval’s Theorem, and sometimes goes by that name as well.
3.4 Convolution

Convolution is an operation on a pair of functions returning a single function in the same variable, defined as follows.

\[(f \ast g)(t) = \int_{-\infty}^{\infty} f(u) g(t-u) \, du.\]

Given the definition above, you can prove the following statement:

\[\mathcal{F}[f \ast g] = \sqrt{2\pi} \mathcal{F}[f] \mathcal{F}[g] \]

3.5 Gaussian distribution

The Gaussian distribution, also called the normal distribution, is more realistic (less idealized) than the Dirac delta distribution. Indeed, the Central Limit Theorem states that samples of independent random variables tend towards a Gaussian distribution.

The Gaussian function centered at \(t = 0\), and with standard deviation \(\sigma\), is given by

\[f(t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{t^2}{2\sigma^2}} \tag{7}\]

The numerical prefactor is fixed so that the function can be interpreted as a probability distribution, i.e. the total area under the curve is 1:

\[\int_{-\infty}^{\infty} f(t) \, dt = 1. \tag{8}\]

Its Fourier transform is given by

\[\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2\sigma^2}{2}} \tag{9}\]

which is another Gaussian shape, with standard deviation \(1/\sigma\). Here \(\tilde{f}(\omega)\) is not normalized to have unit area, but we recall that it is the shape of a Fourier transform that is physically significant, rather than its normalization. (Any numerical factor can be compensated in the formula of the inverse Fourier Transform.)

We have not discussed how to perform the integrals in eqs. (8) and (9), but you can use these formulas to deduce other relations and transforms of Gaussian integrals using basic calculus manipulations.

4 Discrete Fourier Transform

Suppose that we have sampled a function \(f(t)\) at the regularly spaced points \(t_k = \frac{2\pi k}{N}\), for \(k = 0, 1, \ldots, N-1\). We have measured the values

\[f(t_k) = f_k, \tag{10}\]

and now we wish to interpolate among these points with a function of the form

\[f(t) = \sum_{m=0}^{N-1} c_m e^{int}. \tag{11}\]

What values of the coefficients \(c_m\) will reproduce the measured samples? The solution is

\[c_m = \frac{1}{N} \sum_{k=0}^{N-1} e^{-\frac{2\pi i mk}{N}} f_k. \tag{12}\]

This solution can be obtained by observing that the discrete Fourier modes

\[\vec{e}_j = \left(e^{0}, e^{\frac{2\pi i}{N}}, e^{\frac{4\pi i}{N}}, \ldots, e^{\frac{2\pi i(N-1)}{N}}\right), \quad \text{for } j = 0, 1, \ldots, N - 1,\]
which are vectors in $\mathbb{C}^N$, are orthogonal with respect to the inner product defined by

$$\langle \vec{e}_j | \vec{e}_k \rangle = (\vec{e}_j)^* \cdot \vec{e}_k.$$ 

Indeed, we can evaluate the inner product $\langle \vec{e}_j | \vec{e}_k \rangle$ by explicitly calculating that

$$\sum_{m=0}^{N-1} e^{\frac{2\pi im(k-j)}{N}} = \begin{cases} 1 - e^{2\pi i(k-j)} & \text{if } k \neq j, \text{ using the formula for a finite geometric series} \\ 0 & \text{because } k - j \text{ is an integer} \end{cases}$$

In the case $k = j$, each term of the series above is simply 1, so the sum is $N$. A formula valid for either of these cases is therefore

$$\langle \vec{e}_j | \vec{e}_k \rangle = N \delta_{jk}.$$ 

Equation (12) can be interpreted as matrix multiplication, with the DFT matrix

$$a_{mk} = \frac{1}{N} e^{-\frac{2\pi imk}{N}}.$$ 

Because there is a way to implement this operation very efficiently for large $N$, called the “Fast Fourier Transform,” this transform is used in many different contexts.

4.1 Parseval’s Theorem for DFT

With the definitions given above, we can conclude from the orthogonality of the Fourier modes that

$$\frac{1}{N} \sum_{k=0}^{N-1} |f_k|^2 = \sum_{m=0}^{N-1} |c_m|^2.$$