MA22S3 Tutorial Sheet 8: Solutions

29 November – 1 December 2017

1. Find the general solution to

\[ x^2 g''(x) + 3xg'(x) + 3g(x) = x^6 - x. \]

**Solution:**

First step: find two linearly independent complementary solutions to the homogeneous equation

\[ x^2 g''(x) + 3xg'(x) + 3g(x) = 0. \]

Since this is an Euler-Cauchy equation, use the trial function

\[ g = x^m. \]

The characteristic equation is

\[ m(m - 1) + 3m + 3 = 0, \]

which has roots at

\[ m = -1 - \sqrt{2}i. \]

So the two independent complementary solutions can be written as

\[ g_1 = \frac{1}{x} \cos(\sqrt{2} \log x), \quad g_2 = \frac{1}{x} \sin(\sqrt{2} \log x). \]

Second step: find a particular solution to the nonhomogeneous equation. The trial function has the same powers as appears in the nonhomogeneous part, with undetermined coefficients:

\[ g_p = Ax^6 + Bx. \]

The first and second derivatives of this trial function are

\[ g_p' = 6Ax^5 + B, \]

\[ g_p'' = 30Ax^4. \]

We substitute these functions into the differential equation:

\[ 51Ax^6 + 6Bx = x^6 - x. \]

Therefore \( A = 1/51 \) and \( B = -1/6 \), and the particular solution is

\[ g_p = x^6/51 - x/6. \]

Third step: combine the complementary and particular solutions to construct the general solution of the differential equation. It is

\[ g = \frac{c_1}{x} \cos(\sqrt{2} \log x)c_1 + \frac{c_2}{x} \sin(\sqrt{2} \log x) + \frac{x^6}{51} - \frac{x}{6}. \]

2. Solve the following initial value problem.

\[ y''(x) - 6y'(x) + 8y(x) = 9xe^x \]

\[ y(0) = 0 \]

\[ y'(0) = 0 \]

**Solution:**

First step: find two linearly independent complementary solutions to the homogeneous equation

\[ y''(x) - 6y'(x) + 8y(x) = 0. \]

Since this is an equation with constant coefficients, use the trial function
\( y = e^{\lambda x} \). The characteristic equation is \( \lambda^2 - 6\lambda + 8 = 0 \), which has roots at \( \lambda = 2 \) and \( \lambda = 4 \). So the two independent complementary solutions are

\[
y_1 = e^{2x}, \quad y_2 = e^{4x}.
\]

Second step: find a particular solution to the nonhomogeneous equation. The trial function is a product of the trial types for \( x \) and \( e^x \), which is the product of a linear polynomial and the exponential \( e^x \):

\[
y_p = (Ax + B)e^x.
\]

(There is no need to put an extra coefficient in front of \( e^x \), since it can just be absorbed into the unknowns \( A \) and \( B \).)

The first and second derivatives of this trial function are

\[
\begin{align*}
y'_p &= Axe^x + Ae^x + Be^x \\
y''_p &= Axe^x + 2Ae^x + Be^x
\end{align*}
\]

We substitute these functions into the differential equation:

\[
3Axe^x + (-4A + 3B)e^x = 9xe^x
\]

Equating coefficients of like terms, we find

\[
\begin{align*}
3A &= 9 \\
-4A + 3B &= 0
\end{align*}
\]

from which we conclude that \( A = 3, B = 4 \).

Thus the particular solution we have found is

\[
y_p = (3x + 4)e^x.
\]

Third step: combine the complementary and particular solutions to construct the general solution of the differential equation. It is

\[
y = c_1e^{2x} + c_2e^{4x} + 3xe^x + 4e^x.
\]

Fourth step: fix the remaining constants using the initial values. We need the first derivative of \( y \), which is

\[
y' = 2c_1e^{2x} + 4c_2e^{4x} + 3xe^x + 7e^x.
\]

Now we evaluate \( y(0) \) and \( y'(0) \) and set them equal to the values given.

\[
\begin{align*}
c_1 + c_2 + 4 &= 0 \\
2c_1 + 4c_2 + 7 &= 0
\end{align*}
\]

We can solve this linear system of equations by routine elimination to find \( c_1 = -9/2, c_2 = 1/2 \).

Therefore the full solution is given by

\[
y = -\frac{9}{2}e^{2x} + \frac{1}{2}e^{4x} + 3xe^x + 4e^x.
\]