Discrete Fourier Transform:
A function \( f(t) \) has been sampled at the regularly spaced points \( t_k = 2\pi k/N \), for \( k = 0, 1, \ldots, N - 1 \), and the values

\[ f(t_k) = f_k, \]

have been measured.

It is possible to write a function interpolating among these points that takes the form

\[ f(t) = \sum_{m=0}^{N-1} c_m e^{imt}. \]

where the coefficients \( c_m \) are given by

\[ c_m = \frac{1}{N} \sum_{k=0}^{N-1} e^{-\frac{2\pi imk}{N}} f_k. \]

Questions:

1. Suppose we wish to perform a Discrete Fourier Transform for a function that has been sampled at 4 points, with the values

\[
\begin{align*}
f(0) &= 10 \\
f\left(\frac{\pi}{2}\right) &= -5 \\
f(\pi) &= 8 \\
f\left(\frac{3\pi}{2}\right) &= -7
\end{align*}
\]

Compute the function given by the Discrete Fourier Transform, and identify the dominant frequency mode.

Solution:
The DFT matrix for \( N = 4 \) is given by

\[
(a)_{mk} = \frac{1}{4} e^{-\frac{2\pi imk}{4}}, \quad m, k = 0, 1, 2, 3
\]

\[
= \frac{1}{4} \begin{pmatrix}
e^0 & e^0 & e^0 & e^0 \\
e^0 & e^{-\frac{\pi i}{4}} & e^{-\pi i} & e^{-\frac{3\pi i}{4}} \\
e^0 & e^{-\pi i} & e^{-2\pi i} & e^{-3\pi i} \\
e^0 & e^{-\frac{3\pi i}{4}} & e^{-3\pi i} & e^{-\frac{9\pi i}{4}}
\end{pmatrix}
\]

\[
= \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{pmatrix}
\]
Therefore
\[ c_m = \sum_{k=0}^{3} (a)_{mk} f_k \]
\[ = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & i \\ 1 & -i & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 10 \\ -5 \\ 8 \\ -7 \end{pmatrix} \]
\[ = \frac{1}{2} \begin{pmatrix} 3 \\ 1 - i \\ 15 \\ 1 + i \end{pmatrix} \]

The interpolating function is
\[ f(t) = \sum_{m=0}^{3} c_m e^{imt} \]
\[ = \frac{3}{2} + \frac{1 - i}{2} e^{it} + \frac{15}{2} e^{2it} + \frac{1 + i}{2} e^{3it} \]

and the dominant frequency mode obtained from this sampling is the one with the largest value of \(|c_m|\). Here this is the mode with frequency \(m = 2\) and period \(\pi\).

2. Evaluate the following integrals, and show your work.

(a) \( \int_{0}^{3} \delta(x + 2) \ x^2 \ dx = 0 \).
   
   The support of the delta function is at \(x = -2\), which lies outside the range of integration.

(b) \( \int_{0}^{3} (\delta(x + 2) + \delta(x - 2)) \ x^2 \ dx = 4 \).
   
   The first term is zero for the same reason as above. In the second term, the support of the delta function is at \(x = 2\), and the result is \(2^2 = 4\).

(c) \( \int_{-\infty}^{\infty} \delta\left(\frac{x}{4}\right) (x + 3) \ dx = \int_{-\infty}^{\infty} (4u + 3) \delta(u) \ du = 12 \).
   
   (Change of variables: \(u = x/4\))

(d) \( \int_{-1}^{1} \left( \frac{\delta(x)}{dx} \right) \sin x \ dx = \delta(x)\sin x\big|_{-1}^{1} - \int_{-1}^{1} \delta(x) \cos(x) \ dx = -1 \).
   
   (Integration by parts)

(e) \( \int_{-\infty}^{\infty} \delta(x - 1) \delta(x + 1) \ dx = 0 \)
   
   The integrand is definitely zero away from the two points \(x = \pm 1\). One way to see that the integral is zero is to break up the integration interval. For example,
   \[ \int_{-\infty}^{\infty} \delta(x - 1) \delta(x + 1) \ dx = \int_{-\infty}^{-1.5} \delta(x - 1) \delta(x + 1) \ dx + \int_{-1.5}^{-0.5} \delta(x - 1) \delta(x + 1) \ dx + \int_{-0.5}^{0.5} \delta(x - 1) \delta(x + 1) \ dx + \int_{0.5}^{1.5} \delta(x - 1) \delta(x + 1) \ dx + \int_{1.5}^{\infty} \delta(x - 1) \delta(x + 1) \ dx \]
   
   and each of the remaining integrals is 0 by the usual rules of delta function integration.

(f) \( \int_{-\infty}^{\infty} \delta(x^2 - 1)(x + 3) \ dx = \frac{1+3}{1+3} + \frac{-1+3}{1+3} = 3 \)
   
   There are two roots of the equation \(x^2 - 1\). For each of them, we substitute the root into the rest of the integrand and divide by \(\left| \frac{dx}{dx} (x^2 - 1) \right|\).