Continued fractions and $\mathcal{N} = 4$ BPS counting

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Introduction

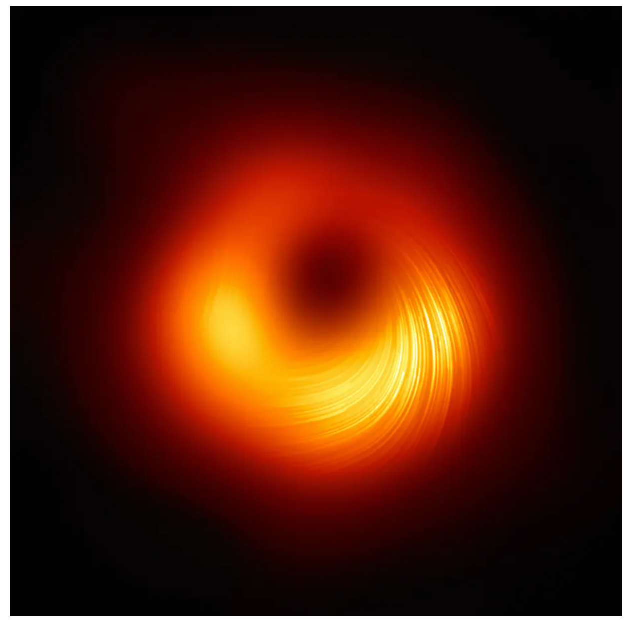
Goal: Study the microscopic degeneracies of negative discriminant $1/4-$BPS dyons, $d(m,n,\ell)_{\Delta<0}$, in $\mathcal{N}=4$ supersymmetric String Theory.

Result: $d(m,n,\ell)_{\Delta<0}$ are encoded in the continued fraction of $\ell/2m$.

Motivation: ~ Modularity implies that $d(m,n,\ell)_{\Delta>0}$ can be obtained from $d(m,n,\ell)_{\Delta<0}$

$$S_{stat}(Q) = \ln d(Q) \qquad \leftrightarrow \qquad S_{BH}(Q)$$

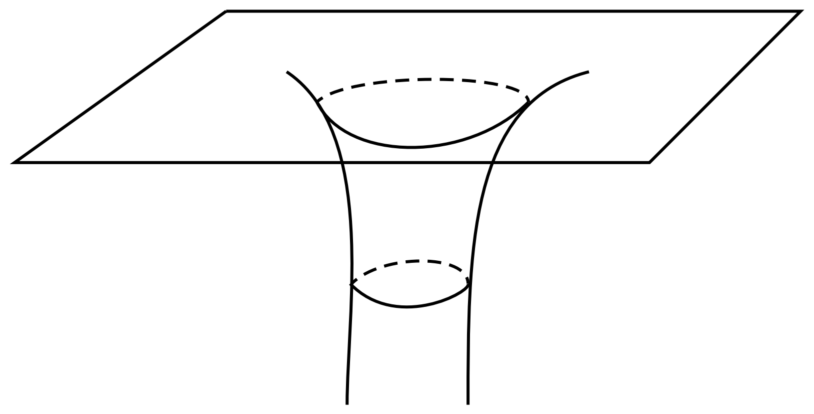
Microscopic | Macroscopic



Inspired by [Chowdhury, Kidambi, Murthy, Reys, Wrase '19]

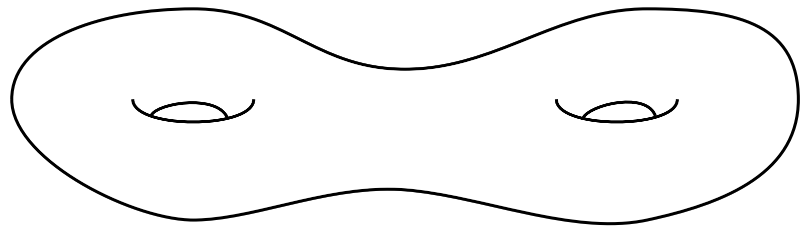
Introduction

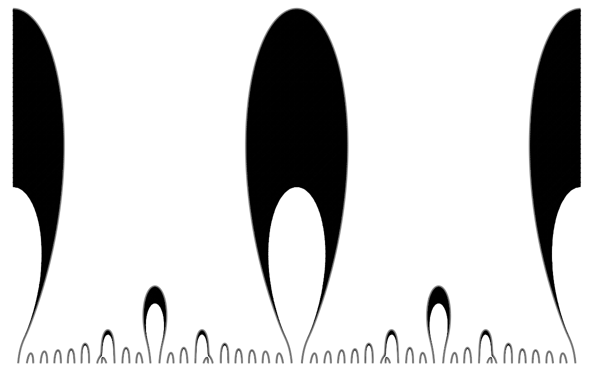
Dyonic degeneracies



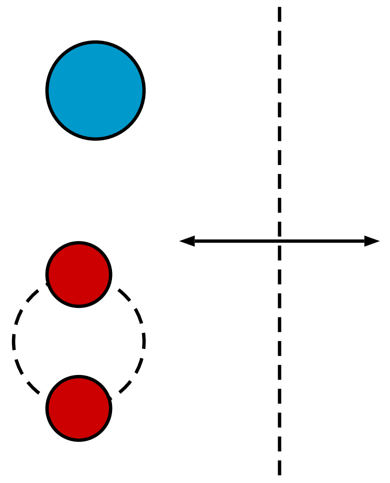
Siegel modular forms

Mock Jacobi forms





Wall-crossing



Continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Setup

Heterotic string theory on T^6

$S-$duality group is $SL(2,\mathbb{Z})$

$\mathcal{N} = 4$ supersymmetry and

28 $U(1)$ gauge groups

$1/4$-BPS states carry electric $\overrightarrow{Q}$ and magnetic $\overrightarrow{P}$ charges: **Dyons**

States characterized by $m = P^2/2 \in \mathbb{Z}$, $n = Q^2/2 \in \mathbb{Z}$, $\ell = P \cdot Q \in \mathbb{Z}$

$$d(\overrightarrow{P}, \overrightarrow{Q}) = d(m, n, \ell)$$

Relevant $S-$duality invariant:

$$\Delta = Q^2 P^2 - (Q \cdot P)^2 = 4mn - \ell^2$$

Area $\sim \sqrt{\Delta}$

Dyon spectrum

Two types of $1/4-$BPS dyons:

Single centered **Immortal**

Two-centered bound states of $1/2$-BPS constituents **Can decay**

[Cheng, Verlinde ’07]

Single centre $1/4$-BPS black holes with finite horizon area have $\Delta > 0$.

We will focus on

$$\Delta = 4mn - \ell^2 < 0$$

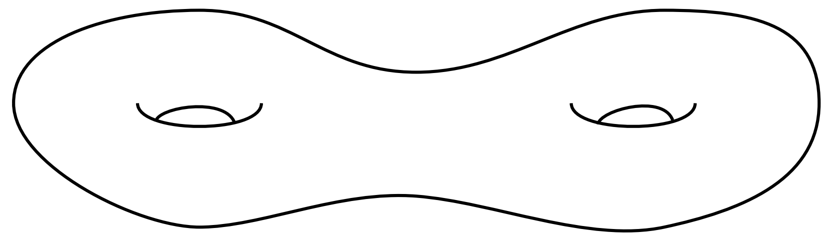
which are always two-centred states

Siegel modular forms

The generating function for $1/4-$BPS dyonic degeneracies is a modular form of the genus-2 modular group $Sp(2,\mathbb{Z})$

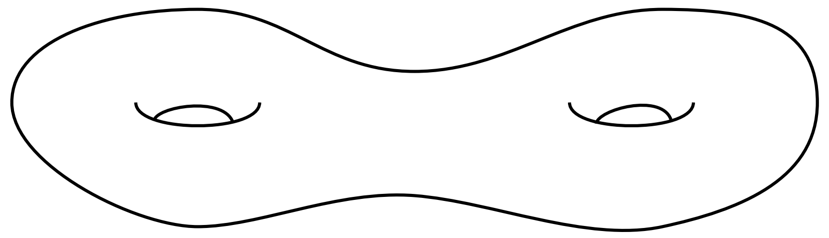
[Dijgkraaf, Verlinde, Verlinde '96]

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{\substack{m,n\geq -1 \\ m,n,\ell\in\mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho+n\sigma+\ell v)}$$



Φ_{10} is the Igusa cusp form, invariant under $SL(2,\mathbb{Z})$.

Siegel modular forms



Extract the degeneracies

$$d(m, n, \ell) = (-1)^{\ell+1} \int_C d\rho d\sigma dv\, p^{-m} q^{-n} y^{-\ell} \frac{1}{\Phi_{10}(\rho, \sigma, v)}$$

$C : 0 \leq \rho_1, \sigma_1, v_1 \leq 1$

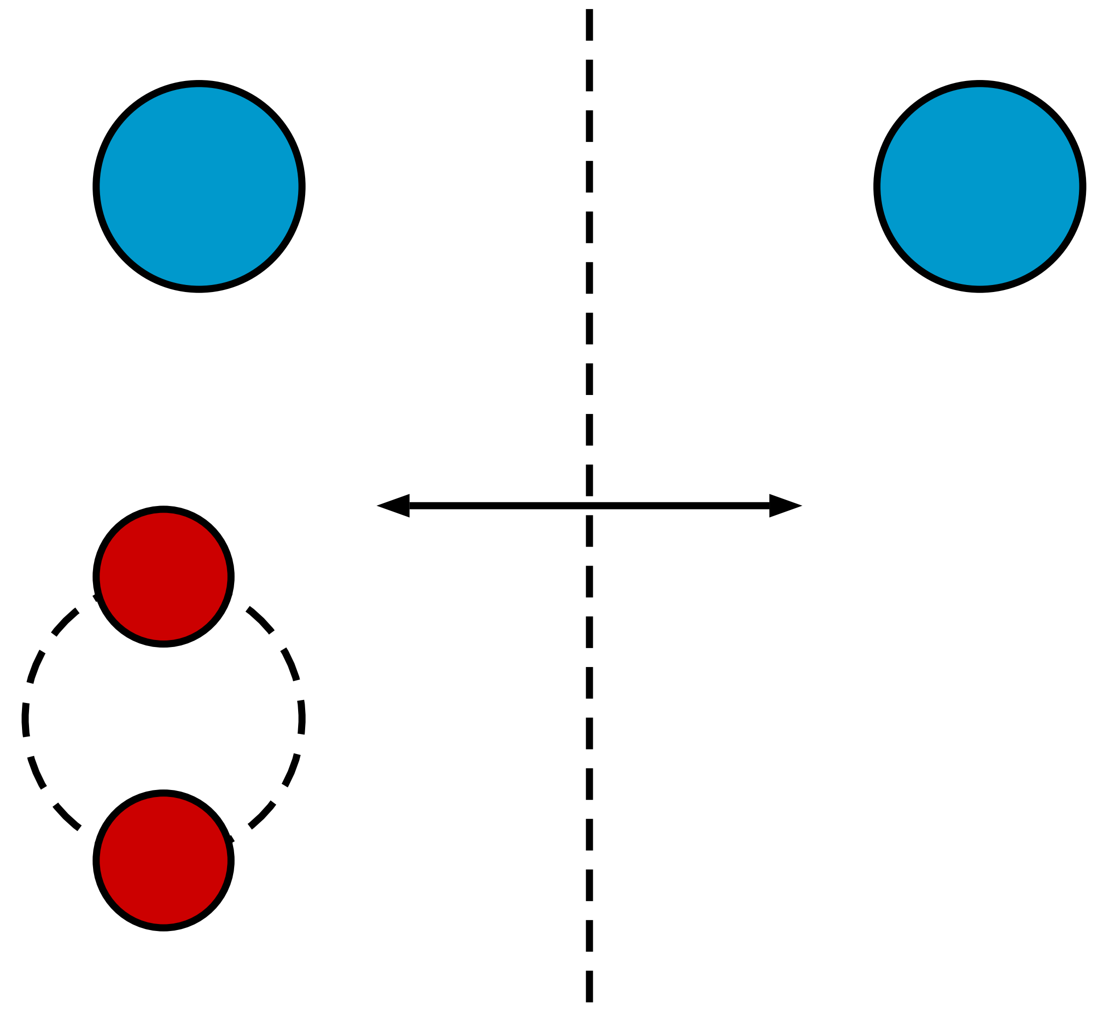
ρ_2, σ_2, v_2 fixed, $\rho_2\sigma_2 - v_2^2 \gg 0$

$$p = e^{2\pi i \rho}$$
$$q = e^{2\pi i \sigma}$$
$$y = e^{2\pi i v}$$

Problem: Meromorphic

Wall-crossing

Pole in the Siegel modular form ⟷



$$d(m,n,\ell) = (-1)^{\ell+1}\int_C d\rho d\sigma dv\, p^{-m}q^{-n}y^{-\ell}\Phi_{10}^{-1}$$

$$C : 0 \le \rho_1, \sigma_1, v_1 \le 1$$

ρ_2, σ_2, v_2 fixed, $\rho_2\sigma_2 - v_2^2 \gg 0$

Changing ρ_2, σ_2, v_2 in contour C

$d(m,n,\ell)$ can jump

[Sen, ’07]

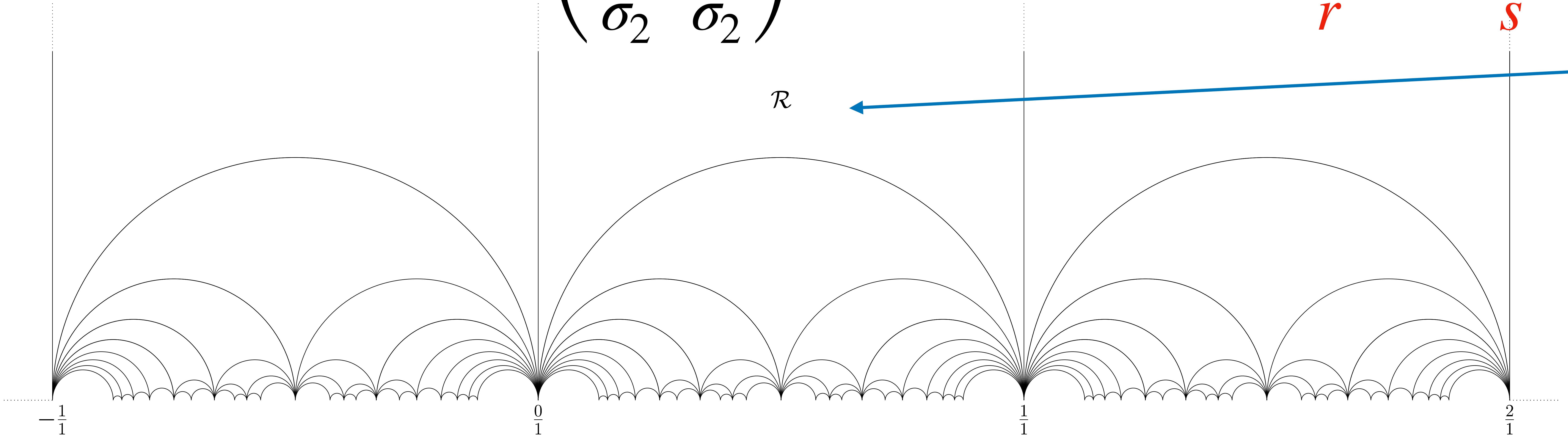
[Dabholkar, Gaiotto Nampuri ’07]

Poles and walls

$\dfrac{1}{\Phi_{10}}$ has an infinite family of second order poles in the (ρ, σ, v) space

$$pq\sigma_2 + rs\rho_2 + (ps + qr)v_2 = 0, \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z})$$

Represent walls in the $\left(\dfrac{v_2}{\sigma_2}, \dfrac{\rho_2}{\sigma_2}\right)$ plane by lines joining $\dfrac{p}{r}$ and $\dfrac{q}{s}$



$\mathscr{R}$-chamber

[Sen, '07]

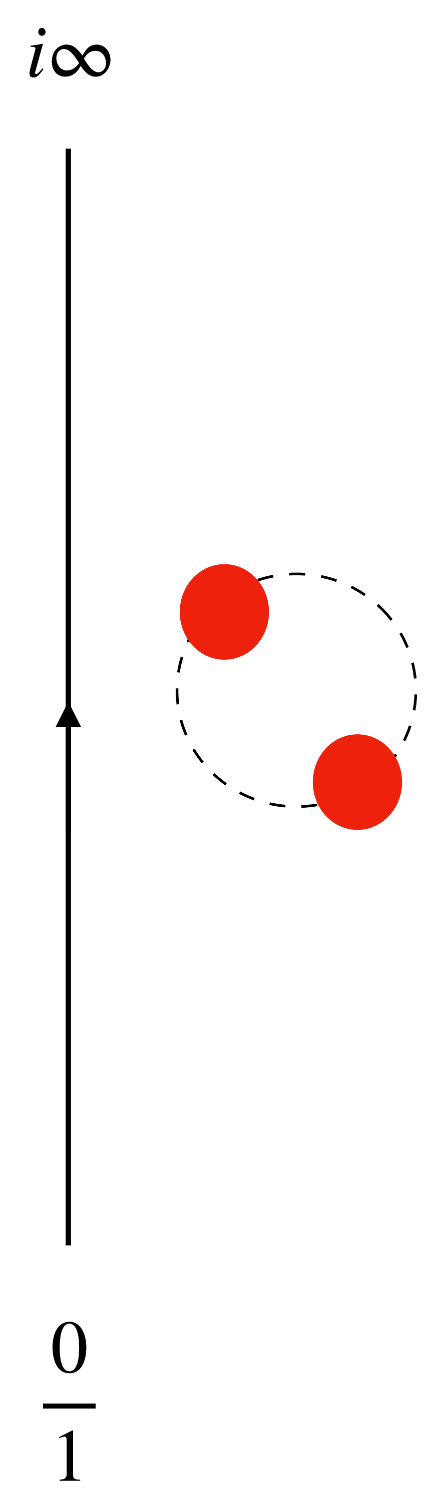
Dyonic decay

From the limit
$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} \xrightarrow{v\to 0} \frac{1}{v^2}\,\frac{1}{\eta^{24}(\rho)}\,\frac{1}{\eta^{24}(\sigma)}$$

the contribution at the pole $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$ is

$$\Delta_\gamma\, d(m,n,\ell) = (-1)^{\ell_\gamma+1}\,|\,\ell_\gamma\,|\,d(m_\gamma)\,d(n_\gamma)\ .$$

where $\dfrac{1}{\eta^{24}(\rho)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n\rho}$



[Sen, '07]

Dyon counting problem

$\Delta < 0 \Longrightarrow$ Two centred-states only

The solution must have the form

[Sen, '11]
[Chowdhury, Kidambi, Murthy, Reys, Wrase '19]

$$d(m,n,\ell) = \sum_{i=1}^{k} \Delta_i = (-1)^{\ell+1} \sum_{\substack{i=1 \\ \gamma_i \in W(m,n,\ell)}}^{k} |\ell_{\gamma_i}|\, d(m_{\gamma_i})\, d(n_{\gamma_i})$$

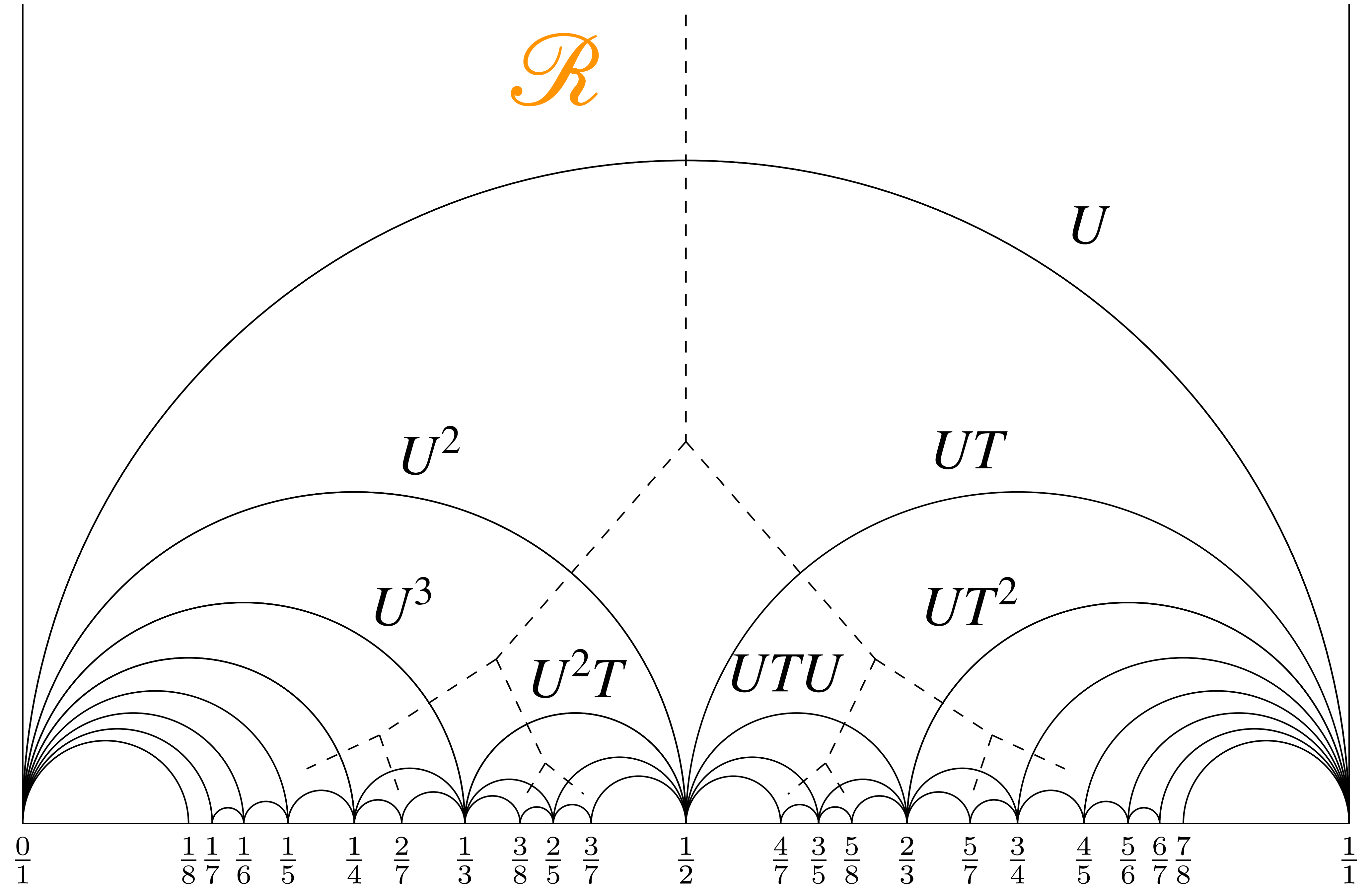
Q: How can we characterize $W(m,n,\ell)$?

Solution

Downwards:

left-right choice associated to

$$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



$$W(m,n,\ell) = \left\{U, U^2, \ldots, U^{s_1}, U^{s_1}T, \ldots, U^{s_1}T^{s_2}, U^{s_1}T^{s_2}U, \ldots, U^{s_1}T^{s_2}U^{s_3}, \ldots, \gamma_*\right\}$$

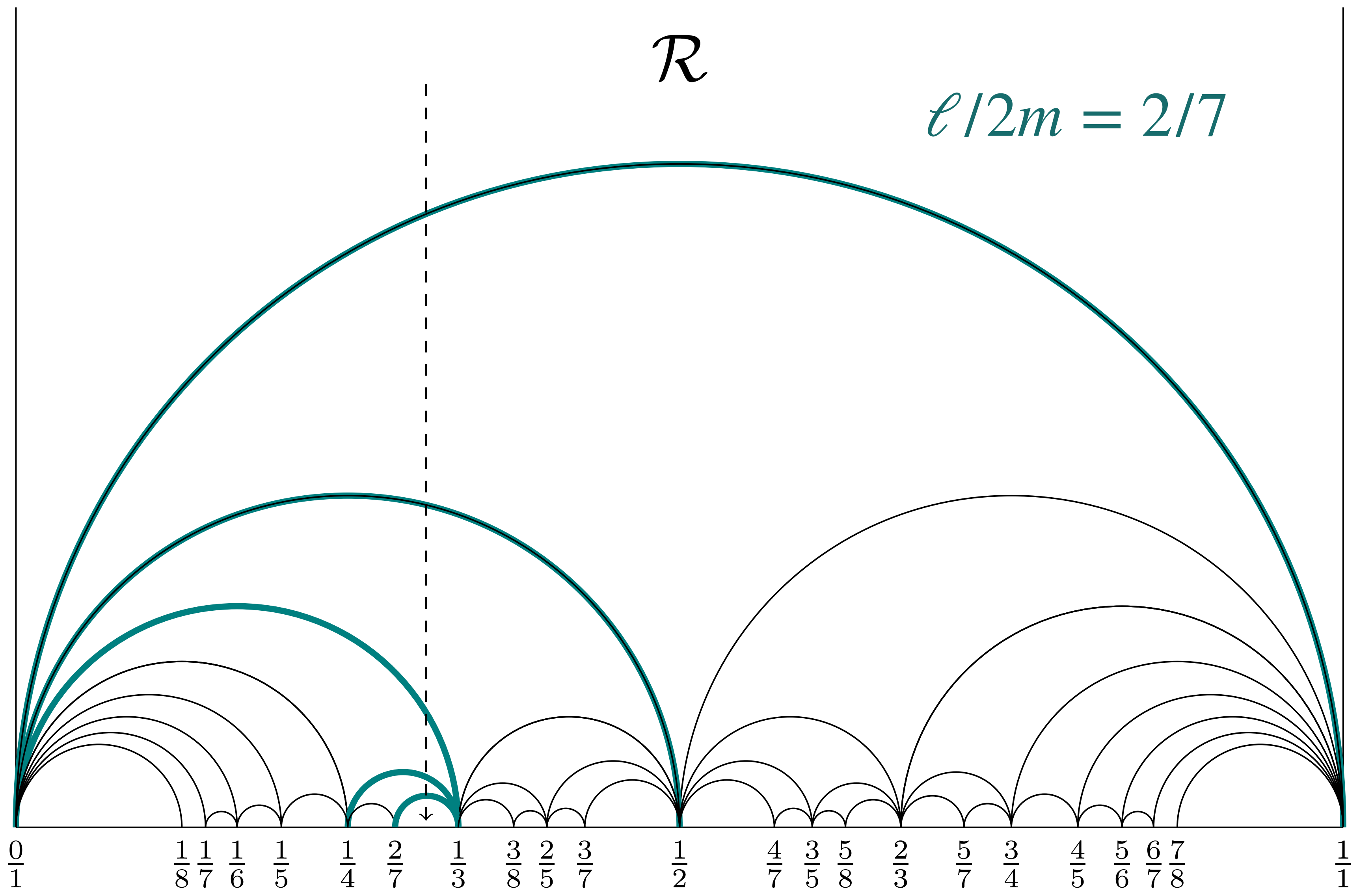
γ_* determines all s_i: Only need to determine γ_*

The continued fraction of $\dfrac{\ell}{2m} = [a_0; a_1, \ldots, a_r] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_r}}}}$ yields

$$\gamma_* = \begin{pmatrix} \ell/g & q \\ 2m/g & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_r & 1 \end{pmatrix}$$

and γ_* satisfies the conditions

$$\frac{\ell}{2m} - \frac{\sqrt{-\Delta}}{2m} < \frac{p}{r} < \frac{\ell}{2m} + \frac{\sqrt{-\Delta}}{2m} \qquad 0 \le \frac{\ell}{2m} - \frac{q}{s} \le \frac{1}{rs}$$



Result

Given m, n, ℓ with $\Delta = 4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$,

$$\ell/2m = [a_0, a_1, \ldots, a_r] \quad \text{defines} \quad W(m,n,\ell)$$

in the $\mathscr{R}$-chamber,

$$d(m,n,\ell) = d_* + (-1)^{\ell+1} \sum_{\substack{i=1 \\ \gamma_i \in W(m,n,\ell)}}^{k} |\ell_{\gamma_i}|\, d(m_{\gamma_i})\, d(n_{\gamma_i})$$

originally, $d(m,n,\ell) = (-1)^{\ell+1} \int_C d\rho d\sigma dv\, p^{-m} q^{-n} y^{-\ell} \frac{1}{\Phi_{10}(\rho,\sigma,v)}$

Jacobi forms

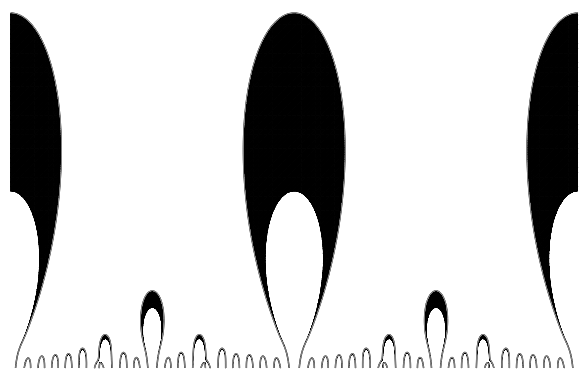
Φ_{10}^{-1} has a Fourier-Jacobi expansion

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m\geq -1} \psi_m(\sigma,v)e^{2\pi i m\rho}$$

where $\psi_m(\sigma,v)$ are Jacobi forms of weight -10 and index m

$$\psi_m\left(\frac{a\sigma+b}{c\sigma+d},\frac{v}{c\sigma+d}\right) = (c\sigma+d)^{-10}\, e^{\frac{2\pi i m c v^2}{c\sigma+d}}\psi_m(\sigma,v)\ ,\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

$$\psi_m(\sigma,v+\lambda\sigma+\mu) = e^{-2\pi i m(\lambda^2\sigma+2\lambda v)}\psi_m(\sigma,v)\ ,\ \lambda,\mu\in\mathbb{Z}$$



Mock Jacobi forms

[Ramanujan ’1920]
[Zwegers ’2001]

$$\psi_m(\sigma, v) = \psi_m^F(\sigma, v) + \psi_m^P(\sigma, v)$$

[Dabholkar, Murthy, Zagier ’12]

split into **mock** Jacobi forms: a finite and a polar part.

$$\psi_m^F(\sigma, v) = \sum_{n,\ell} c_m^F(n, \ell) q^n y^\ell$$ has no poles in (σ, v) **Immortal**

Modularity can be restored at the expense of holomorphicity.

In $\mathscr{R}$, for $0 \le \ell < 2m$,

$$\boxed{d(m, n, \ell) = (-1)^{\ell+1} c_m^F(n, \ell)}$$

Generalized Rademacher expansion

$$
\underset{4mn-\ell^2>0}{c^{\mathrm{F}}_m(n,\ell)} = 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c^{\mathrm{F}}_m(\widetilde{n},\widetilde{\ell}) \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right)
$$

$$
+ \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl\big(\frac{\Delta}{4m},-1;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \qquad \text{(A.12)}
$$

$$
- \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times
$$

$$
\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\, \mathrm{d}u\,,
$$

[Ferrari, Reys, '17]

computes the coefficients $c_m^F(n,\ell)$ with $\Delta > 0$ in terms of $c_m^F(n',\ell')$ with $\Delta < 0$.

Extra: CHL models $N > 1$

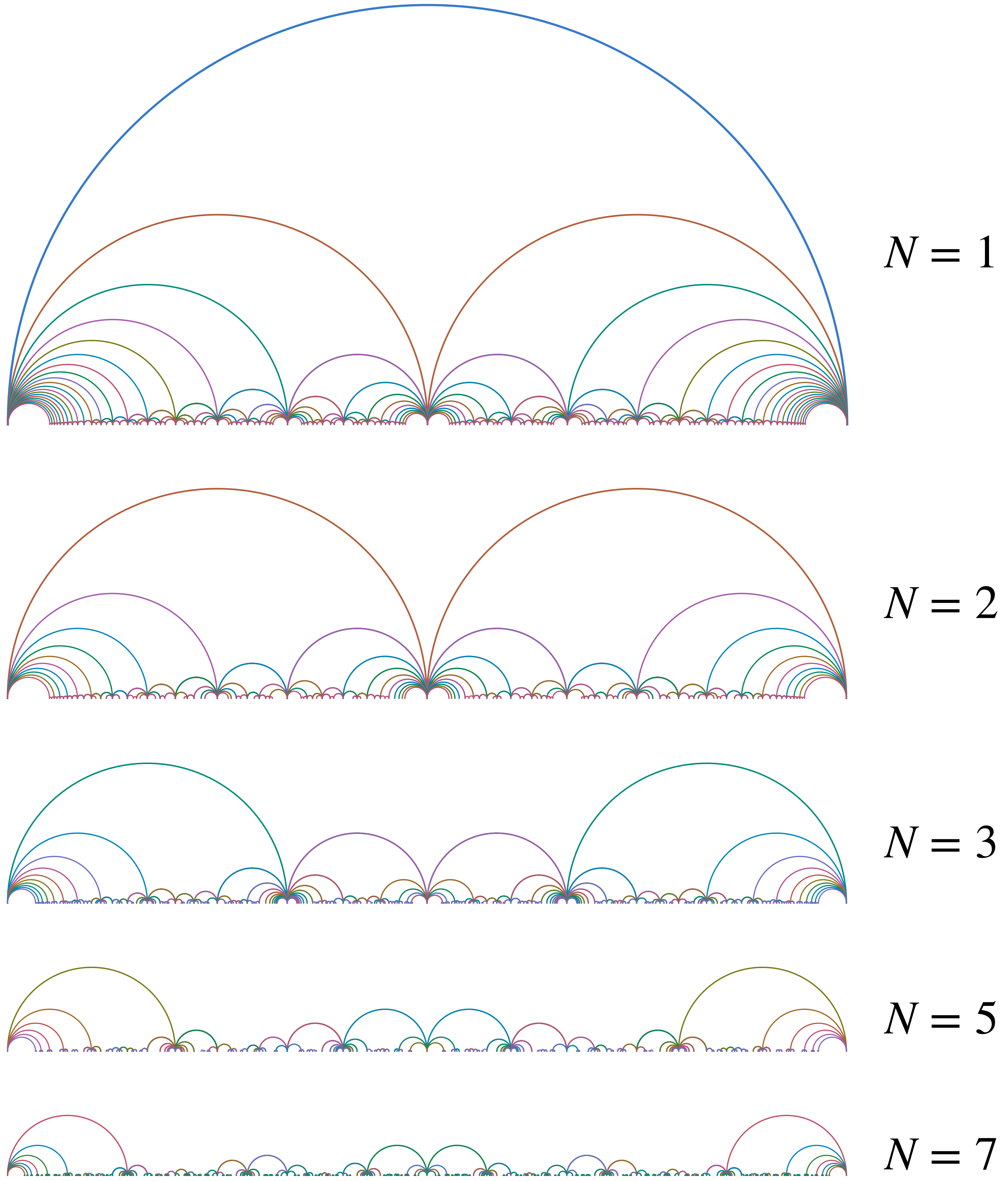
Heterotic string theory on $T^5 \times S^1/\mathbb{Z}_N$ with $N = 2,3,5,7$

Generating functions $\Phi_k(\rho, \sigma, v)^{-1}$. The poles

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(N)$$

The logic is the same, but the details more intricate.

Proceed as earlier, build set $W(m, n, \ell)$ from the continued fraction of $\ell/2m$ but now select the matrices in $\Gamma_0(N)$.



Thank you