Spectrum of kdV charges and thermalization

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Eigenstate Thermalization Hypothesis (ETH)

- What is the set conditions on probe observable O for it to look thermal in a generic state ψ
- ETH tells us that we should look at it's matrix elements in the energy eigenstate:

$$\langle E_i|O|E_j\rangle = \delta_{ij} f_O(E_i) + e^{\frac{-S}{2}} g_O(E_i - E_j) R_{ij}$$

- Then expectations values of O at late times t are given by the canonical ensemble

$$\langle\psi(t)|O|\psi(t)\rangle = Z^{-1} Tr\ O\ e^{-\beta H}$$

[Srednicki '94, Deutsch '91]

Generalized Eigenstate Thermalization Hypothesis (GETH)

- We can ask the same question when the system has infinitely many mutually commuting conserved charges Q_{2n-1}
- What are the conditions that O must satisfy to equlibrate to GGE at late times?

$$\langle\psi(t)|O|\psi(t)\rangle = Z_G^{-1}\ Tr\left(O\ e^{-\sum_n \mu_{2n-1} Q_{2n-1}}\right)$$

Study matrix elements in **mutual eigenstates** of Q_{2n-1} which we call $|E_i\rangle$

- Generalized ETH criterion

$$\langle E_i|O|E_j\rangle = \delta_{ij} f_O\Big(Q_1(E_i), Q_3(E_i), \ldots\Big)$$

[Cassidy, Clark, Rigol '11], [Dymarsky, Pavlenko '19], [Cardy, Calabrese '16]

qKdV Hierarchy in 2d CFT's

- In any integrable 2d CFT, you can construct an infinite set of mutually commuting conserved charges
- classical kdV hierarchy

$$Q_1^{cl} = \int d\phi u(\phi)\,, \quad Q_3^{cl} = \int d\phi u(\phi)^2\,,$$

- Quantum kdV hierarchy

$$Q_1 = \int d\phi T, \quad Q_3 = \int d\phi : T^2 :,$$

qKdV Hierarchy in 2d CFT's

- These charges give us flows in phase space

$$\dot{u} = \{Q_1^{cl}, u\}, \quad \dot{u} = \{Q_3^{cl}, u\} = 6u\partial u - \partial^3 u$$

- Quantum version

$$\dot{T} = [Q_1, T],$$
$$\dot{T} = \{Q_3, T\} = -3\partial(TT) - \frac{c-1}{6}\partial^3 T$$

- In a seminal work, the existence and relation to integrability was shown

$$[Q_{2k-1}, Q_{2l-1}] = 0$$

[Bazhanov, Lukyanov, Zamalodchikov '96]

Form of KdV charges

- Expressions for these charges are not known.
  They are known in terms of Virasoro generators only for Q_3, Q_5 and Q_7
- They look like:

$$Q_1 = L_0 - \frac{c}{24}$$

$$Q_3 = L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2880} + 2\sum_{n=1}^{\infty} L_{-n} L_n$$

Preferred basis

- The states $L_{-m_1}...L_{-m_k}|\Delta\rangle$ form a basis of the Verma module
- Q_{2n-1} keep us within the Verma module
- There is a particular basis in the Verma module which is eigenbasis of qKdV charges

$$
\begin{aligned}
|\psi\rangle &= L_{-m_1}...L_{m_k}|\Delta\rangle + ... \\
Q_{2n-1}|\psi\rangle &= \lambda_{2n-1}|\psi\rangle
\end{aligned}
$$

Lessons about quantum kdV spectrum

[AK, Dymarsky, Sugishita, to appear]

- n_k is defined in the free boson representation of the CFT: n_k count the number of times k appears in the set $\{m_i\}$

$$|\{n_k\}, \Delta\rangle = |\{m_i\}, \Delta\rangle$$

- Example

$$L_{-2}^2 L_{-1} \quad \text{is} \quad n_2 = 2, \quad n_1 = 1$$

- Structure of quantum kdV spectrum:

$$Q_{2n-1} = \Delta^n + c^{n-1} \sum_k n_k \; f_1(k, \Delta) + c^{n-2} \left(\sum_k n_k^2 \; f_2(k, \Delta) + \sum_{k,p} n_k n_p \; g_1(k, p, \Delta) \right) + O(c^{n-3})$$

Explicit forms of quantum kdV spectrum

- changed variables: $\tilde{\Delta} = \Delta - \frac{c-1}{24}$, $\tilde{n}_k = k + \frac{1}{2}$

$$
\begin{aligned}
Q_{2n-1} = \tilde{\Delta}^n + \sum_k \sum_{j=0}^{n-1} & \frac{(2n-1)\,\Gamma\left(n+1\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(n-j\right)} \\
& \left(\frac{c}{24}\right)^j \tilde{\Delta}^{n-1-j} k^{2j+1} \tilde{n}_k \\
& + c^{n-2}\left(\sum_k n_k^2 f_2(k,\Delta) + \sum_{k,p} n_k n_p g_1(k,p,\Delta)\right) \\
& + O(c^{n-3})
\end{aligned}
$$

- Obtained closed form expressions for f_2 and g_1 as well.

Broad strategy

- We will first calculate the classical KdV charges Q^{cl}_{2n-1}
- Large c expansion in the quantum theory is equivalent to expansion action variables I_k in the classical theory.

$$Q^{cl}_{2n-1} = h^n + \sum_k f_1(k) I_k + \sum_k f_2(k) I_k^2 + \ldots$$

- **Semi-classical quantization rule:**:
Multiply Q^{cl}_{2n-1} by $\left(\frac{c}{24}\right)^n$ and

$$I_k \longrightarrow \frac{24}{c}\left(n_k + \frac{1}{2}\right), \quad h \longrightarrow \frac{24}{c}\left(\Delta + \frac{c}{24}\right)$$

- **Constraint from Modular covariance**

$$\langle Q_{2n-1} \rangle_\beta = \text{modular covariant with weight 2n}$$

How to understand the expansion of Q^{cl}_{2n-1}? Integrability and Lax Pairs

- A pair of operators L, M such that

$$\frac{d}{dt}L = [M, L]$$

  is equivalent to equations of motion.
- This is useful because they generate all the integrals of motion

$$I_k = TrL^k$$

- These quantities are then automatically conserved by cyclicity of trace

$$\frac{d}{dt}I_k = TrL^{k-1}[M, L] = 0$$

- Trace over spectrum of L^k gives us action variables

[Lax '68]

- An observation by Lax about the kdV equation
  Defining
  $$L = -\frac{d^2}{dx^2} + u(x)$$
  and
  $$M = -4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + u\frac{d}{dx}u\right)$$
- $$\dot{L} = [M, L]$$
  is equivalent to kdV equation:
  $$\dot{u} = 6uu' - 4u'''$$

Novikovs method

- Schrodinger type equation:

$$-\frac{d^2}{dx^2}\psi + \frac{u}{4}\psi = \lambda\psi$$

- Iso-spectral deformations of u are generated by the kdV generators Q_{2k-1}

$$\delta u = \frac{c}{24}\{Q_{2k-1}, u\}$$

- Example

$$\dot{u} = \frac{c}{24}\{Q_3, u\} = 6uu' - 4u'''$$

Novikovs method

[Novikov '74]

- To study solutions $u(x)$ of

$$\frac{c}{24}\{Q_{2k-1}, u\} = 0$$

  Study the spectral problem of

$$-\frac{d^2}{dx^2}\psi + u\psi = \lambda\psi$$

- **Inverse scattering problem**: Given spectrum of the Schrodinger equation
- Try and reconstruct the potential $u(x)$
- This problem was solved by Novikov for periodic $u(x)$.

Turn Novikovs analysis Perturbative

[Novikov '74]

- Perform the appropriate phase space integrals perturbatively

$$I_k = \frac{i}{\pi} \oint_{a_i} dp \log \lambda$$

- The conserved quantities

$$Q_{2n-1} = -\frac{\Gamma(n+1)(\Gamma(1/2))}{(\Gamma(n+1/2))(2\pi i)} \oint_{\infty} dp \lambda^{n-1/2}$$

- Our approach: Do it perturbatively in distance between λ_i
- Reduces higher genus phase space integrals to torus ones which are tractable.
- It allows us to get the expansion

$$Q^{cl}_{2n-1} = h^n + \sum_k f_1(k) I_k + \sum_k f_2(k) I_k^2 + \ldots$$

Applications to thermalization: work in progress and future directions

- In the thermodynamic limit, at leading order in c: generalized ETH was shown in [Dymarsky, Pavlenko '19]
- What happens at higher order in c?
- How can we use these results to make universal statements about hydrodynamical degrees of freedom?
- Thank You