# UNIVERSITY OF DUBLIN

2106

## TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

**Foundation Scholarship** **Hilary Term**

COURSE 131/141

Monday 23 March 1992 J.C.R. 09.30 - 12.30

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1. A metal chain of length $L$ and mass $M$ is hung vertically above a surface with one end in contact with it. The chain is then released to fall freely. If $x$ is the distance covered by the end of the chain, how much force (exerted by the bottom surface) will the chain experience at any instant during the process?

2. Suppose a person jumps off a building 202m high onto cushions having a total thickness of 2m. If the cushions are crushed to a thickness of 1/2m, what is the person's acceleration as the person slows down? Assume that the cushions decelerate the person at a constant rate. Express your answer in terms of $g$.

3. Calculate the gravitational force on a particle with mass $m$ located at $P = (O, O, z)$ due to a disk centered at $O$ in the $xy$ plane. Assume a disk with radius $R$ and uniform mass density $\sigma$(in kg/m$^2$).

4. Consider the universe at $t = 0.01$ seconds after the Bang. It was a boiling, expanding, homogeneous soup of radiation and matter. Its expansion is described by Hubble's "constant" H: An observer on a typical particle sees a particle a distance $R$ away moving outward with velocity $v = HR$. Let $\rho$ be the universe's mass density at $t = 0.01$ seconds. By $E = mc^2$, radiation energy density can be converted into an equivalent mass density, and, in fact, $\rho$ is dominated by radiation at these early times.
Find a relation between $\rho, H$ and the gravitation constant $G$ that ensures that the universe will keep on expanding and not collapse on itself. (**Hint:** Consider a typical point 0 in the soup and a typical particle $m$. What is the condition for $m$ to keep on moving away from 0?)

5. A bead of mass $m$ is constrained to move on a hoop of radius $r$. The hoop rotates with constant angular velocity $\omega$ around the vertical axis as shown.

   (a) Consider the inertial frame 0 at the centre of the hoop. 0's axes are **fixed** in space and its $z-$axis coincides with the hoop's axis of rotation. Indicate the forces on $m$ in 0.

   (b) Find the equilibrium positions ($\theta$) of the bead on the hoop for $r\omega^2 \geq g$ and $r\omega^2 < g$. What is $\theta$ for $\omega \to \infty$?

   (c) Consider an accelerating frame $0'$ rotating with the bead whose $z$ axis is parallel to 0's. Indicate the forces on $m$ in $0'$.

   (d) Find the critical angular velocity $\Omega$ below which the bottom of the hoop provides a stable equilibrium position for the bead. That is, if the bead deviates from $\theta = 0$ by a small amount, it will be pulled back. Use $\sin\theta \simeq \theta, \cos\theta \simeq 1$ for $\theta \to 0$. To start this problem find the sum of the forces on $m$ tangential to the hoop, denoted by $F_\theta$. Your answer should agree with part (b).

6. A minimum of a continuous function $f(x)$ of a real variable $x$ can be bracketed by specifying three values $a < b < c$ such that $f(a) > f(b)$ and $f(c) > f(b)$. A measure of the uncertainty in the position of a minimum so bracketted is $|c - a|$ (the width of the interval which brackets the minimum). Describe the Golden Section Search Algorithm which allows one to iteratively reduce the uncertainty in the position of the minimum of $f(x)$ so bracketed. Show that the most efficient implementation of this algorithm reduces the uncertainty in the position of the minimum after each iteration by a factor $(1 - R)$ where

$$R = \frac{3 - \sqrt{5}}{2}$$

7. Consider the differential equation:

$$\frac{d^2y}{dx^2} - y = 0$$

with initial conditions

$$\begin{aligned} y\big|_{x=0} &= 1 \\ \left.\frac{dy}{dx}\right|_{x=0} &= 0 \end{aligned}$$

Rewrite this equation as two coupled first order equations.

Describe Euler's numerical algorithm to integrate these first order equations, and apply this algorithm with a fixed step size of $h = 0.25$ for exactly two steps to find

$$y\big|_{x=0.5} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=0.5}$$

Estimate the error in your numerical solution.

8. (a) Show that

$$(\underline{a} \times \underline{b}).(\underline{c} \times \underline{d}) = (\underline{a}.\underline{c})(\underline{b}.\underline{d}) - (\underline{a}.\underline{d})(\underline{b}.\underline{c})$$

and in particular

$$(\underline{a} \times \underline{b})^2 = \underline{a}^2\underline{b}^2 - (\underline{a}.\underline{b})^2.$$

(b) Show that

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0$$

9. (a) Solve the differential equations

(i)

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

(ii)

$$\frac{dy}{dx} = \frac{2x - y - 5}{x + 2y}$$

(b) Show that the family of curves $xy = a$ is given by the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}$$

Write down and then solve the differential equation for the orthogonal trajectories.

10. Show that the system of linear differential equations:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j$$

$i = 1, .., n$ with $a_{ij}$ constant can be written in matrix form. Define a *mode* of the system and show that it involves an eigenvector and associated eigenvalue.

Find the general solution of the equations:

$$\frac{dx}{dt} = x - 6y$$

$$\frac{dy}{dt} = -6x - 4y$$

11. Establish the formula:

$$\dim \text{Ker } T + \dim \text{Im } T = \dim M$$

where $T : M \to N$ is a linear operator.

Let rank $T$ = rank $T^2$.

Prove that

$$\text{Ker } T \cap \text{Im } T = \{0\}.$$