

Problem Solvers Week 2

1. (Aran Nolan)

Suppose we choose two random numbers a and b and then we raise a to the power of b to get a new number. We add up all of the digits in this number to get a newer number. Then we add up all of the digits in this newer number to get a new newer number. We continue this process until we are left with a number that is just a single digit. What is the probability that digit is 1? For example

73829

29

11

2

2. How can we put 100 apples into two bags such that one bag has twice as many as the other bag. None of the apples can be harmed.
3. Explain the following partition of the nonnegative integers; $A = \{0, 3, 6, 8, 9 \dots\}$, $B = \{1, 4, 7, 11, 14 \dots\}$, $C = \{2, 5, 10, 12 \dots\}$.
What are the relative sizes of these sets?
4. (IV Paper 1992)
Let $a_1, a_2, a_3 \dots$ be the sequence of positive integers that have no "9"s in their decimal representation. Show that this series

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$$

converges.

5. What is the most number of parts we can divide the plane into with n lines?
6. Compute the integral $\int_0^{\frac{\pi}{2}} \cos^2(\cos x) + \sin^2(\sin x) dx$
7. Prove there exist two positive irrational numbers s and t such that s^t is rational.
8. On average how many times must one roll a die until all of the digits have come up at least once?

9. Define a bijection between $(0,1]$ and $(0,1)$.
10. If we start filling in real entries to an $n \times n$ matrix (to any slot), at what point can we stop and guarantee that no matter what remaining entries are filled in, the determinant will be nonzero? Can we beat $\binom{n+1}{2}$?
11. (Emma Howard)
 Suppose we have a bunny placed at some integer along the real line. It takes some integer jump j to the left or to the right. And then continues taking that jump size in the same direction. Every time it takes one jump, we get to drop a bomb somewhere on the number line. Can we hit the bunny with a finite number of bombs?
12. (Tim Murphy)
 There is a rabbit in the middle of a circular pond. A poacher is on the edge of the pond. The poacher can run 4-times as fast as the rabbit can swim. Can the rabbit get away ?
13. $\int_a^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x} dx$
14. Derive the determinant of the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} & a_{n+1}^{n-1} \\ a_1^n & a_2^n & \cdots & a_n^n & a_{n+1}^n \end{pmatrix}$$

without using induction.

15. (Vladimir Dotsenko)
 Show that the system of equations

$$\begin{aligned} z_1 + 2z_2 + \dots + nz_n &= 0, \\ z_1^2 + 2z_2^2 + \dots + nz_n^2 &= 0, \\ &\dots \\ z_1^n + 2z_2^n + \dots + nz_n^n &= 0. \end{aligned}$$

has the only complex solution $z_1 = \dots = z_n = 0$.

16. Alice and Bob play the following game with a monic polynomial with zero coefficient equal to 1; They take it in turns to fill in the coefficients of all the x powers between 2 and $n-1$. If the resulting polynomial has no real root, Alice wins; otherwise, Bob wins. If Alice goes first, who has a winning strategy?