Problem Solvers Week 3

1. (Mark Allen)
   If a rabbit in the centre of a circular pond can swim with speed v, what is the maximum speed a poacher on the edge of the pond could have so that the rabbit can still escape?

2. On average how often does one need to roll a die before a 6 comes up?

3. How likely is it that a drunkard's walk (stumbles left or right with probability half) beginning at 0 on the number line will reach $\pm 2^n$ before returning to 0?

4. (Karl Downey)
   What is the average number of random steps a bug on a vertex of a cube must take to get to the opposite vertex?

5. What's the most likely; Rolling 6 dice and getting 1 or more sixes, rolling 12 dice and getting 2 or more, or rolling 18 dice and getting 3 or more? Explain.

6. Alice and Bob play the following game based on the total roll of two standard dice. Alice says that a 12 will be rolled first. Bob says that two consecutive 7s will be rolled first. They keep rolling until one of them wins. Who is more likely to win?

7. What is the largest number that can be written in standard mathematical notation, using only three decimal digits and no other symbols?
   Hint; Reference the works of James Joyce and Donald Knuth

8. 1, z, z^2, and z^3 (all distinct) lie on the same circle on the complex plane. What is the centre of that circle?

9. Can we rearrange the digits in a power of two to get a different power of two?

10. Prove $\sin x \leq x$ for all x.

11. Give an intuitive proof as to why the derivative of $\sin x$ is $\cos x$.

12. Consider the equation $x^n = e$ in a finite group G. Prove the number of elements that satisfy this equation must be a divisor of the order of the group.

13. The smallest distance between any two of six towns is d kilometres. The largest distance between any two of the towns is D kilometres. Show that $\frac{D}{d} \geq \sqrt{3}$. Assume the land is flat.

14. $\int_0^1 \frac{x^2-1}{\log x} dx$

15. $\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx$

16. $\int_0^1 \int_0^1 \frac{1}{1 - xy} \, \mathrm{d}x \mathrm{d}y$

17. (Sean Diffley) The positive integers are partitioned into two sets:
f(1), f(2), f(3) ...
g(1), g(2), g(3) ...
With f(1) < f(2) < ...
And g(1) < g(2) < ...
And g(n) = f(f(n)) + 1
Find f(240)

18. (Keith Glennon)
Construct an explicit one-parameter family of homotopic curves between the top half and bottom half of the unit circle, and a continuous one-parameter family of Riemannian metrics such that at any fixed value of the parameter the homotopic curve is a geodesic for that metric