Complex Analysis Questions

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1 Introduction

This is an attempt to create a list of all possible "give definition" or "prove theorem" type questions that can be asked based on the notes for the course so far. It includes all exercise/problem type questions given so far in lectures, but doesn't include any homework exercises (yet). The emphasis is on providing a complete overview of what has been covered in the course so far, in chronological order, as well as an easy way to test how well the material is understood, and so I make no attempt to mimic realistic 'exam' questions, nor do I attempt to divide the questions into easy and hard. For the type of questions you'll actually see on an exam, see one of the several other lists of questions available.

I intend to make one of these for at least each of the Maths courses, and possibly for the Physics classes too.

1.1 Week 1

The first week covered the basic definitions of complex numbers and sets. Everything here should be pretty obvious, with the possible exception of one or two of the proofs.

1. Define the field of complex numbers.
2. Verify the field axioms for the field of complex numbers.
3. Define the *modulus*.
4. Define the *complex conjugate*.
5. Let $z, w \in \mathbb{C}$ Show that the following hold:
    (a) triangle inequality;

(b) reverse triangle inequality;

(c) parallelogram law.

6. Show how each $z = x + iy \in \mathbb{C}$ can be expressed in

   (a) Trigonometric form;

   (b) Polar form;

7. Show how the product of two complex numbers is equivalent to adding the angles and multiplying the lengths.

8. Prove de Moivre's formula

9. Solve the following:

   (a) $z^3 = 1$

   (b) $z^4 = 16i$

10. Define the *open disk.*

11. Define an *interior point.*

12. Define a *boundary point.*

13. Define an *isolated point.*

14. Describe the points in the following sets:

    (a) $A = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$

    (b) $A = \{\frac{i}{n} : n \in \mathbb{N}\}$

15. Define an *open set*

16. Define a *closed set.*

17. Define a *bounded set.*

18. Define a *compact set.*

19. State whether the following are open, closed, bounded or compact.

    (a) $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) \leq 1, \mathrm{Im}(z) = 0\}$

    (b) $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1, \mathrm{Im}(z) = 0\}$

20. Define a *connected set.*

21. Is the set $\{z \in \mathbb{C} : |z| = 1 \text{ and } \mathrm{Im}(z) \neq 0\}$ connected?

22. Define a *convergent sequence*.

23. Define the *Cauchy criterion*.

24. What two statements are equivalent to "(z_n) is convergent"

25. Prove this.

26. Show that:

    (a) $\left(\frac{i^n}{n}\right)_{n=1}^{\infty}$ converges to 0

    (b) $\left(\cos\frac{1}{n} + i\sin\frac{1}{n}\right)_{n=1}^{\infty}$ converges to 1

27. Define a *limit point*

28. Prove the alternative definition for a closed set:

    Theorem 1.1 *Let $A \subseteq \mathbb{C}$ Then A is closed if and only if A contains all of its limit points.*

29. Define the *limit* of a function.

30. Prove the following theorem:

    Theorem 1.2 *Let $f, g : A \longrightarrow \mathbb{C}$ with*

    $$L = \lim_{z\to w} f(z) \text{ and } M = \lim_{z\to w} g(z)$$

    then:

    $$\begin{aligned} L &= \lim_{z\to w} f(z) + g(z) = M + L \\ L &= \lim_{z\to w} f(z)g(z) = ML \\ L &= \lim_{z\to w} \frac{f}{g}(z) = \frac{M}{L}, L \neq 0 \end{aligned}$$

31. Define *continuity* at a point.

1.2 Week 2

This week mostly focused on the continuity and differentiability of complex functions.

1. What two statements are equivalent to 'f is continuous on A' (for an open set A in $\mathbb{C}$)

2. Prove this.

3. Prove the following theorem:

   Theorem 1.3 *Let $A, B \in \mathbb{C}$ be open sets and $f : A \to \mathbb{C}$, $g : B \to \mathbb{C}$ both continuous with $f(a) \subset B$. Then $g \circ f : A \to \mathbb{C}$ is continuous.*

4. Define *complex differentiability.*

5. Show that the function

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z \end{aligned}$$

   is complex differentiable everywhere in $\mathbb{C}$.

6. Show that the function

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \bar{z} \end{aligned}$$

   is not complex differentiable everywhere in $\mathbb{C}$.

7. Prove that if $f : A \longrightarrow \mathbb{C}$ is complex differentiable at $w \in A$ then f is continuous at w.

8. Prove the sum, product and quotient rules for complex differentiable functions $f, g : A \longrightarrow \mathbb{C}$ at $w \in A$.

9. Prove the following theorem:

   Theorem 1.4 *Let $A \subseteq \mathbb{C}$ be an open set,*

$$z_0 = x_0 + iy_0 \in A$$

$$f : A \longrightarrow \mathbb{C}\ ,\ f = u + iy$$

   Then f is complex differentiable at z_0 if and only if:

- *u and v are both real differentiable functions at* (x_0, y_0)
- *The Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ,\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  are satisfied at (x_0, y_0)

1.3 Week 3

This week we proved the complex chain rule, defined analytic functions and started looking at the exponential, logarithmic, and trigonometric functions.

1. Define and prove the complex chain rule.
2. Define an *analytic function*
3. Show that

$$\begin{array}{rcl} \mathrm{Exp} : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z = x + iy & \longmapsto & e^x(\cos y + i \sin y) \end{array}$$

   is analytic in $\mathbb{C}$.
4. Why are polynomials analytic in $\mathbb{C}$?
5. Why are rational functions analytic on their rational domain?
6. Define the exponential function.
7. How does the exponential function map complex numbers geometrically?
8. Prove that the exponential function is analytic in $\mathbb{C}$.
9. Show that $e^{z+w} = e^z e^w$
10. Prove that $e^z \neq 0\ \forall z \in \mathbb{C}$.
11. Show that the exponential function is periodic with period $2\pi i$.
12. Define complex sin and cos.
13. Why are sin and cos analytic in $\mathbb{C}$?
14. Show $\frac{d}{dz} \cos z = -\sin z$ and $\frac{d}{dz} \sin z = \cos z$.

15. Prove the following theorem:

    Theorem 1.5 *Let $f :\longrightarrow \mathbb{C}$ be an open, one-to-one, complex differentiable function at $z_0 \in A$.*
    Let $g : f(A) \longrightarrow \mathbb{C}$ be the inverse of f, and suppose g is continuous.
    If $f'(z_0) \neq 0$, then g is complex differentiable at $w_0 := f(z_0)$, with $g'(w_0) = \frac{1}{f'(z_0)}$.

16. Define the complex *logarithm function.*

17. Define the *principal logarithm.*

18. Prove that the logarithm function Log is analytic on $\mathbb{C}\backslash(-\infty, 0]$ (and state why it is not analytic on $\mathbb{C}\backslash\{0\}$).

19. Show that $\mathrm{Log}'(w_0) = \frac{1}{w_0}$.

20. Define the complex *square root.*

21. Give an expression for the various values of $\sqrt{w}$.

22. Show that $\sqrt{w}$ is analytic on $\mathbb{C}\backslash(-\infty, 0)$.

23. Show that $\sqrt{w}$ is continuous on $\mathbb{C}\backslash(-\infty, 0)$.

24. At which points is

    $$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z = x + iy &\longmapsto x^2 + iy^2 \end{aligned}$$

    complex differentiable?