An Intuitive Approach to the Residue Theorem

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Introduction

The point of this document is to explain how the calculation of residues and the Residue Theorem works in an intuitive manner. In it I explain why we calculate the residues as we do, and why we can compute the integrals of closed paths by simply summing the residues of the singularities contained within. If, like me, you found the fully rigorous presentation of these topics in the class difficult to follow and to understand intuitively, you may find this approach useful. The inspiration for this document was from reading Roger Penrose's *Road to Reality* where he discusses the multi-valuedness of the complex logarithm. I'd recommend the book to anyone actually interested in theoretical physics, it's great.

Complex Integrals

As we know, if a complex function f is analytic (holomorphic), i.e. complex differentiable at every point in the domain, then, supposing γ is some path connecting a and b, the integral of the function over the path is given by

$$\int_\gamma f(z)dz = F(b) - F(a)$$

where $F(z)$ is the anti-derivative of $f(z)$. That is, it doesn't matter what path we take from a to b, the integral is the same.

So for instance, the integral of $f(z) = z$ from 1 to 2 is going to be $2^2/2 - 1^2/2 = 3/2$ since the anti-derivative of $f(z) = z$ is $F(z) = z^2/2$.

It should be obvious then, that the integral of a closed path (i.e. one where the end points are the same, a full loop) should be 0, as we have $F(b) - F(b) = 0$.

We can also think of this intuitively as continuously deforming the path, as we are allowed to, by shrinking the curve until eventually it shrinks down to nothing and the integral is 0.

Singularities

However, something weird happens if our function contains a singularity, both for functions over closed paths, and for open paths. Suppose we have a path as in the first image in Figure 1. We cannot smoothly deform our function through

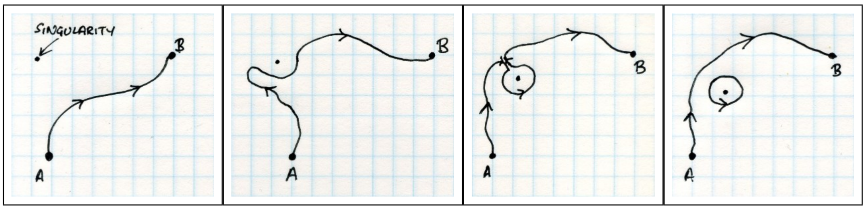

Figure 1: Deforming a path

the singularity since the function isn't defined there, in a sense we 'get caught on it' as if our curve was a rope getting snagged on a nail. However, we can deform our curve like in the middle images in Figure 1, and then cancel out the sections of the curve that follow an identical but opposite path so we are left with the final image.

We can make the loop around our singularity smaller and smaller without changing the value of the integral, however it's clear that no matter how small we make it, we can't make it disappear altogether. Just like we can tie a rope around an incredibly thin lamp post, but no matter how tight we make it the knot won't suddenly disappear.

If our loop is a closed loop, then, if it contains a singularity, clearly we can't shrink it until it disappears completely, so it should be clear that a closed integral around a singularity isn't going to be zero like before. Similarily, the integral in the final image in Figure 1 is going to differ somehow from the integral in the first image.

Since we can shrink our loop as much as we want, but we can't make it disappear altogether, it's clear that whatever's going on with the singularity is going on at the limit as we approach the singularity. So let's examine our function 'near' the singularity.

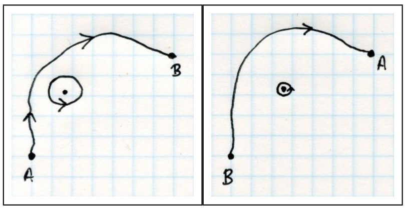

Figure 2: We look at what happens as we make the loop infinitesimal

Suppose that the singularity is at a point c. Just like a Taylor Series expansion for real functions, we can do a Laurent Series expansion about c for our complex function:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \ldots + a_{-2}\left(\frac{1}{z-c}\right)^2 + a_{-1}\left(\frac{1}{z-c}\right) + a_0 + a_1(z-c) + a_2(z-c)^2 + \ldots$$

We wish to integrate this function over a closed loop. We consider each of the terms in the expansion as a function in itself and look at the integral of each of these over the path, and view the original function as the sum of all of these. This will greatly simplify analysing the behaviour.

Positive terms

So, looking first at a_0, $a_1(z-c)$ and terms of higher order, it should be obvious that there's going to be no singularity at any of the points, since we're looking at functions like z, z^2 etc. and there are no points on the complex plane where these aren't defined and therefore integrating any of these around a closed loop is going to be zero. So we can ignore these terms completely for the final integral.

Negative terms

Now, let's look at the a_{-2} term. We want to integrate $1/(z-c)^2$ over a loop around c. Intuitively we can see that the radius of our loop is going to be $r = |z-c|$, while if z is very small (i.e. $z \ll c$) then $1/(z-c)^2$ is going to be approximately $1/c^2$ at all points of the loop, and because the value of the function is approximately the same at all points on our loop, we can simplify computing our integral[1] by just multiplying our circumference $2\pi r$ by the approximate value to get the final value for our integral,

$$a_{-2} 2\pi r/c^2.$$

Then, as r approaches zero, the integral approaches zero as well. So we can ignore the a_{-2} term in our function, and similar reasoning allows us to ignore the a_{-3} term and all lower terms down to negative infinity.

The important term!

Now, we have seen we can ignore all terms except the $a_{-1}1/(z-c)$ term. Let's look at that one now. Using the same reasoning as above, our loop is going to have radius $|z-c|$. The value of our function at any point z on our curve will be $a_{-1}/(z-c)$. Since we're looking at 'small' loops, we'll just multiply this value by the circumference of our loop, since this will work 'in the limit'. So:

$$\frac{a_{-1}}{z-c} \cdot \underbrace{|z-c|2\pi}_{\text{circumference}} = 2\pi a_{-1} \cdot \frac{|z-c|}{z-c}$$

You may think then that as our loop shrinks our integral approaches $2\pi a_{-1}$, but unfortunately, while our hand-wavy argument was enough to banish all the other terms, we actually need to do this integral properly to get the right answer.

Integrating 1/z properly

For simplicity, we look at the function $f(z) = 1/z$. To integrate this function we parameterise the curve using a function

$$\gamma(t) = e^{2\pi i t} \qquad 0 \leq t \leq 1.$$

Suppose $F(z) = F(\gamma(t)) = (F \circ \gamma)(t)$ is the anti-derivative of our function. In that case,

$$\int_\gamma f(z)dz = \int_a^b (F \circ \gamma)'(t)dt$$

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[1] If this is unconvincing, feel free to prove this more rigorously with limits or whatever

is our integral, and by the chain rule $(f \circ g)'(x) = f'(g(x))g'(x)$ this becomes

$$\int_a^b f(\gamma(t))\gamma'(t)dt$$

which is the familiar formula for complex integration which we have just derived. Now, $f(\gamma(t)) = 1/e^{2\pi it}$, and $\gamma'(t) = 2\pi ie^{2\pi it}$, so our integration simply becomes

$$2\pi i \int_0^1 dt = 2\pi i$$

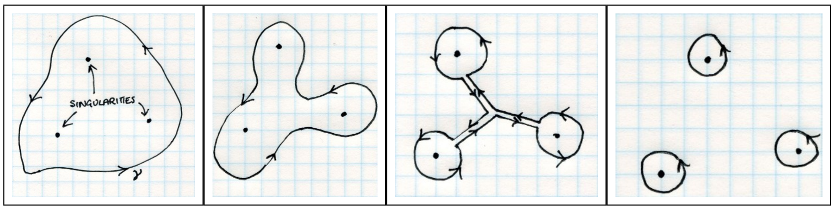

Figure 3: Closed loop around multiple singularities

Summary

So integrating a closed loop around the function f(z) $= a_{-1}1/z$ gives us the result $a_{-1}2\pi i$. And by extension, integrating a closed loop around any sensible function will give us $a_{-1}2\pi i$ since we have seen how any of the other terms in the Laurent expansion can be ignored.

Of course if we have a closed loop around a function with multiple singularities, then we can simply add up the results of the integration of a number of singularities, and if our loop goes around them multiple times, or in a different direction, then this can be taken into account in the obvious way. This leads us to our definition:

The Residue Theorem

- Define $\mathrm{Ind}_\gamma(z)$ as the *winding number* of the path γ about the point z.
- Define the *residue* of a function f and a point z_0 as the coefficient a_{-1} in the Laurent Series
$$f(z) = \sum_{n=-\infty}^{\infty} a_k(z - z_0)^k$$
and denote it $\mathrm{Res}(f, z_0)$.

In that case, the residue theorem states that, given an open and star shaped set A in $\mathbb{C}$, and an analytic function $f : \setminus\{z_1, \ldots, z_n\} \to \mathbb{C}$ with isolated singularities at $\{z_1, \ldots, z_n\}$, and a piecewise C^1 closed path γ in $\{z_1, \ldots, z_n\}$, that

$$\int_\gamma f(z)dz = 2\pi i \left[\sum_{j=1}^{N} \mathrm{Res}(f, z_j)\mathrm{Ind}_\gamma(z_j)\right]$$

Hopefully, my argument above means that the residue theorem is as obvious as $e^{2\pi i} = 1$! Note that my argument breaks down at the origin, although the residue theorem *does* hold for singularities at the origin. However, it should be enough to convince you of the truth of the theorem in general.

Addendum: Complex Logarithm

We have the complex logarithm defined as

$$log(z) = \{log|z| + i\mathrm{Arg}(z) + 2\pi ni : n \in \mathbb{Z}\}.$$

Unlike the usual definition of the real logarithm, the complex logarithm is multi-valued. This shouldn't really be surprising, the logarithm tells us what power we must raise e to in order to get a desired result, and as we know, adding $2\pi ni$ to any exponent does not change the result, i.e.

$$e^z = e^{z+2\pi ni} \quad n \in \mathbb{N}.$$

We can look at this as being essentially a full 'complex rotation' of the exponent. However, there is another way of looking at the complex logarithm, using residues. Note that $\frac{d}{dz}\log(z) = \frac{1}{z}$ so:

$$log(z) = log(z) - log(1) = \int_1^z \frac{1}{z} dz$$

Of course, as we have seen the function $1/z$ has a singularity, so we can understand intuitively the multi-valuedness of the complex logarithm as our ability to deform the path of the integral from 1 to z to make as many windings around 0 as we like, and since the residue about 0 is $2\pi i$, this is the difference between the various values of $\log(z)$!