Physics Scholarship Examination Questions

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Contents

1 Thermodynamics

1. TODO: 2012 Q 1
2. TODO: 2012 Q 2

3. Scholarship Exam 2011 Q.5.

   - (3 marks) Starting from the central law of thermodynamics, derive the first Energy Equation

   $$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial U}{\partial V}\right)_V - P$$

   - (2 marks) Interpret the physical meaning of the a and b constants that distinguish the van der Walls equation of state from the ideal gas equation of state:

   $$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

   - (2 marks) Show that unlike the ideal gas, U for the van der Waals gas does depend on more than T.
   - Show that, however, C_V for a van der Waals gas depends only on T.

   Hint: Assume U and C_V depend on both T and V and evaluate the differentials for each.

4. Scholarship Exam 2011 Q.6.

   Air condition a car:

   - (3 marks) Draw a PV diagram of a general Carnot refrigeration cycle. Assume the coolant to be an ideal gas, show that the ratio of heat exchanges in the Carnot cycle is given by the ratio of temperatures of the thermal reservoirs.
   - (4 marks) A car air conditioner operates as a Carnot cycle refrigerator between outside temperature T_h and a lower temperature interior T_l. The car naturally gains heat from the outdoors at a rate $A(T_h - T_l)$ by conduction; this heat is the one to be removed by the air conditioner. For a steady state condition, show that the power P required to maintain the car at temperature T_l is given by

     $$P = A\left(T_l - 2T_h + \frac{T_h^2}{T_l}\right)$$

     Hint: heats can be evaluated by taking the process over some arbitrary time interval Δt.
   - (1 mark) If the outdoors is at 40 degC and the car is to be maintained at 21 degC against a heat gain coefficient of $A = 16,288.1$ W/K from the outside, find the power P required by the air conditioner. (Note that an average small car engine can produce up to a maximum 100 kW power).
   - (2 marks) If the sun adds an additional amount of heating power P_{sun} to the car interior through radiation, how would the equation above be modified to account for this?

5. Scholarship Exam 2010 Physics Q.5.

   - (10 marks) Show that the difference between isothermal and adiabatic compressibilities is

     $$\kappa_T - \kappa_S = \frac{T\beta^2 V}{C_P}$$

     where β is the volume expansivity. You will need to use a Maxwell relation. You may use the relation

     $$C_P = T\left(\frac{\partial S}{\partial T}\right)_P$$

6. Scholarship Exam 2010 Physics Q.6

   Consider two identical bodies of *constant* heat capacity C_P and with negligible thermal expansion coefficients.

   - (2 marks) Show that when they are placed in thermal contact in an adiabatic enclosure, their final temperature is $T_f = (T_1+T_2)/2$ where T_1 and T_2 are their initial temperatures.
   - (3 marks) Now consider these two bodies being brought to thermal equilibrium by a Carnot engine operating between them. The size of the cycle is small so that the temperature of the bodies do not change appreciably during one cycle. Show that the final temperature is $T_f = \sqrt{T_1 T_2}$.
   - (5 marks) Show that the entropy increase for the process of the first part of the question above is given by

     $$\Delta S = 2C_P \ln \frac{T_1 + T_2}{2\sqrt{T_1 T_2}}$$

   State your methods and assumptions clearly

7. Scholarship Exam 2009 Physics 2 Q.5.

   Imagine summer in a hot place and a room air conditioner.

   - (4 marks) Draw a PV diagram of a general Carnot heat engine. Assuming the working substance to be an ideal gas, show that the ratio of heat exchanges in the Carnot cycle is given by the ratio of temperatures of the thermal reservoirs:

   $$\frac{Q_l}{Q_h} = \frac{T_l}{T_h}$$

     where Q_l and T_l are the heat exchange and temperature of the colder reservoir respectively, and Q_h and T_h are the heat exchange and temperature of the hotter reservoir respectively.

   - (5 marks) A room air condition operates as a Carnot cycle refrigerator (heat engine in reverse) between inside temperature T_h and a lower temperature T_l. The room gains heat from the outdoors at a rate $A(T_h - T_l)$; this heat is removed by the air conditioner. The power supplied to the cooling unit is P. Show that the steady state temperature of the room is given by

   $$T_l = (T_h + \frac{P}{2A}) - \sqrt{(T_h + \frac{P}{2A})^2 - T_h^2}$$

   - (1 mark) If the outdoors is at 37 degC and the room is maintained at 17 degC by a cooling power of 2 kW, find the heat loss coefficient A of the room in W/K.

8. Scholarship Exam 2009 Physics 2 Q.6.

   Adiabatic compression.

   - (3 marks) Starting with the central law of thermodynamics, derive the 2nd TdS equation

   $$TdS = C_p dT - T\left(\frac{\partial V}{\partial T}\right)_P dP$$

   - (5 marks) A block of metal is subjected to an adiabatic and reversible increase of pressure from P_1 to P_2. Show that the initial and final temperature T_1 and T_2 are related as:

   $$\ln\frac{T_2}{T_1} = \frac{\beta V}{C_P}(P_2 - P_1)$$

     Hint: You can approximate the volume of the block to be constant. β is the volume thermal expansivity, and C_P is the isobaric heat capacity for the metal.

   - (2 marks) If instead the block was compressible, how might this affect the final temperature T_2? Consider the cases of the block floating in space vs. sitting on a bench on earth. Discuss in 50 words or less.

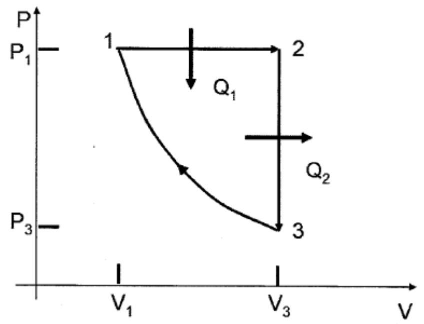
9. Scholarship Exam 2008 Physics 2 Q.1.

A hypothetical engine, with an *ideal gas* as the working substance, operates in the cycle shown below.

- (3 marks) Define the efficiency η of an engine.
- (7 marks) Show that the efficiency of this engine is given by:

$$\eta = 1 - \frac{1}{\gamma}\left[\frac{1 - P_3/P_1}{1 - V_1/V_3}\right]$$

Recall γ is the ratio of heat capacity at constant pressure to the heat capacity at constant volume. Clearly state the assumptions and laws that you use.



10. Scholarship Exam 2008 Physics 2 Q.2.

You are to show that the ratio of isobaric and isochoric heat capacities can be expressed by the ratio of isothermal and adiabatic compressibility for a general PVT system, i.e.

$$\frac{C_P}{C_V} = \frac{\kappa_T}{\kappa_S}$$

- (2 marks) Write down the definitions of the isothermal and adiabatic compressibility.
- (4 marks) Starting with the central equation of thermodynamics, show that

$$\left(\frac{\partial U}{\partial S}\right)_V = T \quad \text{and} \quad C_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

- (4 marks) Use the additional following expression for isobaric heat capacity to derive the required result:

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P$$

11. Scholarship Exam 2007 Physics Paper 2 Q.1.

- (4 marks) Write notes on the definition of *heat* and *entropy*, and the key properties of the latter.
- (4 marks) A piece of metal, of heat capacity C_1 at temperature T_1 is thrown into a liquid filled calorimeter of heat capacity C_2 at temperature T_2. The eventual temperature is T_3. Subsequently the system is vigorously stirred and the temperature rises to T_4. Discuss the entropy changes this system and give a formula for the *total* entropy change.
- (2 marks) What thermodynamic principle does this result illustrate?

12. Scholarship Exam 2007 Physics Paper 2 Q.2.

- (4 marks) Sketch a proof of Carnot's (First) Theorem.
- (4 marks) An engine is designed to operate with four reversible processes at constant pressure or volume only. If the minimum and maximum pressure and volume are p, v and P, V, how much (net) heat is supplied to this system in a single cycle?
- (2 marks) Discuss whether your result is valid for an ideal gas, a real gas, or any fluid.

13. Scholarship Exam 2006 Physics Paper 2 Q.1.

- (3 marks) Write notes on the meaning of "reversibility" in thermodynamics and discuss whether Joule free expansion is reversible.
- (3 marks) An ideal gas is heated from 0 to 100 degrees Celsius at constant volume and then cooled at constant pressure back to 0 degrees. What is the net heat added per mole in this process?
- (4 marks) Draw a diagram illustrating the process described above and a further isothermal one which completes a closed cycle. What heat is involved in this final step? What is the work done *on* the gas in this cycle (arguing directly from the First Law)?

  (All processes are to be considered reversible)

14. Scholarship Exam 2006 Q.2.

- (3 marks) Write brief notes on the definition and essential properties of entropy in thermodynamics.
- (4 marks) Hence develop expressions for C_V and C_P in terms of entropy.
- (3 marks) Use cyclical relations etc. to relate the ratio of these two heat capacities to the ratio of the isothermal and adiabatic bulk moduli.

15. Scholarship Exam 2005 Physics 2 Q.1.

    (Note: no marks indicated for 2005 papers or earlier)

    - Discuss *briefly* the basic relations between thermodynamic quantities that may be taken to define an ideal gas. State and derive a formula for the difference between the two molar heat capacities C_V and C_P of an ideal gas.
    - At room temperature and pressure a typical solid has density three orders of magnitude greater than that of air, a molar heat capacity of the same order as that of a gas, and an expansivity of the order of 10^{-3}/K. Make a detailed semi-quantitative argument to the effect that the difference of its heat capacities is negligibly small for many purposes, by generalising the analysis used above.

16. Scholarship Exam 2005 Physics 2 Q.2.

    - State the Second Law of Thermodynamics in *either* of the two traditional forms: Kelvin-Planck or Clausius.
    - Draw a labelled diagram to illustrate how Carnot's theorem (for Carnot cycles) can be derived from the second law, as stated by you above, with a *brief* indication of the logic of this derivation.
    - A heat pump is sold for operation between 280 and 300K. In addition to the costs of the power supplied, it has associated capital and running costs that exceed those of ordinary resistive electrical heating by an amount which is twice the cost of the power supplied to the latter. Discuss quantitatively whether this is an advantageous purchase.

17. Scholarship Exam 2004 Paper 2 Q.1.

    - Write notes on the theorems of Carnot that relate to ideal heat engines, indicating the manner of their derivation, etc.
    - A sample of gas consisting of one mole, initially at temperature T_0 is subjected to a Joule expansion process increasing its volume from V_0 to $3V_0$. Treating the gas as ideal, what is its final temperature?
    - The expansion is followed by the reversible processes of isobaric compression and heating at constant volume, which returns the gas to its original state. How much work is done by the system in the isobaric process? How much work is done by the system in the entire cycle?

18. Scholarship Exam 2004 Paper 2 Q.2.

    - Derive one of Maxwell's relations from the Central Equation and state the others.
    - Show that heat capacity at constant volume may be related to F according to
      $$C_V = -T \left(\frac{\partial^2 F}{\partial T^2} \right) v$$

2 Optics

1. TODO: 2012 Q 1
2. TODO: 2012 Q 2

3. Scholarship Exam 2011 Q.1.

- (3 marks) Consider a material with complex refractive index $n = n_1 + in_2$. Show that the intensity of light which has been travelling for a distance z through a material, is given by $I = I_0 e^{-4n_2 \pi z/\lambda_0}$
- (2 marks) What is the relationship between relative permittivity and refractive index for a material? Use this relationship to find relationships between both real (n_1) and imaginary (n_2) parts of the refractive index and real (ε_1) and imaginary (ε_2) parts of the relative permittivity (i.e. find $n_1 = f(\varepsilon_1, \varepsilon_2)$ and $n_2 = f(\varepsilon_1, \varepsilon_2)$).
- (3 marks) Derive an approximate expression for n_2, as a function of ω, for a gas (use your physical intuition to estimate n_1).
- (2 marks) What is the maximum value of n_2 for a gas of sodium atoms with an atom density of 10^{17}m^{-3}?

  (The main sodium absorption is at $\lambda = 590$nm and has $\gamma \approx 6 \times 10^8$rad/s)

Note:

$$\varepsilon_1(\omega) = \frac{Ne^2}{\varepsilon_0 m} \frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]} \quad \varepsilon_2(\omega) = \frac{Ne^2}{\varepsilon_0 m} \frac{\gamma\omega}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

4. Scholarship Exam 2011 Q.2.

- (3 marks) Two waves are polarised in the x and y directions with amplitudes E_{0x} and E_{0y} respectively. Both waves have identical wavelength and period. However there is a relative phase shift of ε between the waves. These waves can be described by

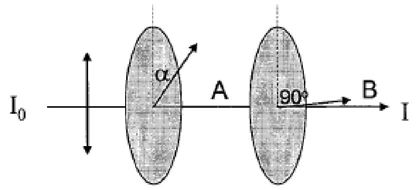
  $$\boldsymbol{E}_x = \boldsymbol{i} E_{0x} \cos(kz - \omega t)$$
  $$\boldsymbol{E}_y = \boldsymbol{j} E_{0y} \cos(kz - \omega t + \varepsilon)$$

  Show that the resultant wave has a range of possible x and y components (E_x and E_y) given by

  $$\left(\frac{E_y}{E_{0y}}\right)^2 + \left(\frac{E_x}{E_{0x}}\right)^2 - 2\left(\frac{E_y}{E_{0y}}\right)\left(\frac{E_x}{E_{0x}}\right)\cos\varepsilon = \sin^2\varepsilon$$

  Note: $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- (2 marks) Explain, using a diagram, what this equation describes.
- (1 mark) In certain optical equipment, it can be important to rotate the polarisation vector of light by $\pi/2$ radians (90deg) One way to do this is to have linearly polarised light passing through a polariser with transmission axis at an angle, α, to the initial direction of polarisation. The light then passes through another polariser with transition axis at right angles to the original direction of polarisation (see figure). The result is that that the direction of polarisation has been rotated by $\pi/2$ radians.

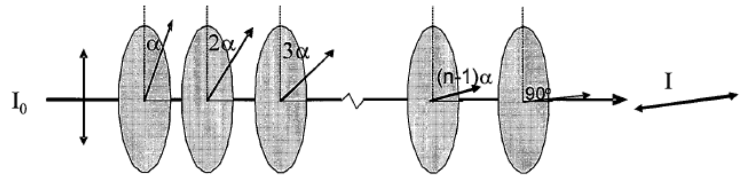
  What is the final intensity of the transmitted light as a fraction of the initial intensity, I_0 and α? (i.e. find $I = f(I_0, \alpha)$).



- (3 marks) It can be shown that the final intensity is maximised if $\alpha = \pi/4$. Then, $I/I_0 = 0.25$. This is a lot of light wasted in order to rotate the polarisation!

  It would be useful to find a way of rotating the polarisation direction by $\pi/2$ radians without losing so much light.

  One possibility would be to imagine an array of n polarisers, each one rotating the polarisation by an angle α (see below)



  Show that the final intensity is given by $I = I_0 \cos^{2n}(\pi/2n)$. What is the final intensity as $n \to \infty$.

- (1 mark) Suggest why this approach won't work in practice.

5. Scholarship Exam 2010 Q.1.

   Consider a lightsource fabricated from a long wire. When a large current is driven through it, the wire heats up and emits light. This can be thought of as a one-dimensional light source. The electric field of the emitted light depends on the radial distance from the wire, r as

$$\boldsymbol{E} = \boldsymbol{E_0}(r)\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

- (4 marks) By considering the power output per unit length, P_L of the light source, derive an expression for E_0 as a function of r.

- (4 marks) An experiment measured the light intensity, $I(r)$ for different values of r:

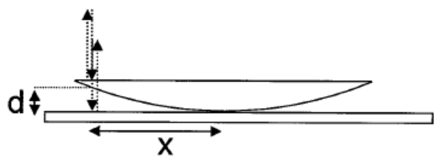
$I(r)$, (W/m^2)	160	80	53	40	32
r, (m)	0.1	0.2	0.3	0.4	0.5

  By writing down an expression relating $I(r)$ to r and by appropriately plotting and fitting the experimental data, find P_L. How much light energy (in Joules) does a 10m long light source emit in an hour?

- (2 marks) Consider a light source fabricated as a very large two-dimensional sheet. Work out how the amplitude of the electric field of the emitted light scales with perpendicular distance, r, from the light source.

6. Scholarship Exam 2010 Q.2.

   - (2 marks) Very briefly explain why it is more general to use phase differences rather than path differences when calculating interference effects.
   - (4 marks) A famous interference pattern is called "Newton's Rings". This is formed when a plano-convex lens sits on a flat glass plate. When viewed from above circular interference fringes are seen.

     This is due to light reflecting from the top surface of the plate interfering with light reflecting from the bottom surface of the lens.

     Find an expression for the phase difference δ between rays incident a distance x from the centre of the lens assuming x is small. (Note that $1 - a)^{1/2} \approx 1 - a/2$ if $a \ll 1$).
   - (2 marks) The light intensity when the rays interefere is given by $I = 4I_0 \cos^2(\delta/2)$, find the light intensity at the centre of the lens.
   - (2 marks) Show that constructive interference for radii given by $x = \sqrt{R\lambda(2m-1)}$ where R is the radius of curvature of the lens.

7. Scholarship Exam 2009 Paper 2 Q.3.

   Consider a wavefront moving in the positive x direction through space. In front of the wavefront, the electric and magnetic fields are zero, $\boldsymbol{B} = \boldsymbol{E} = 0$. However, everywhere behind the wavefront, the electric and magnetic fields are constant, $\boldsymbol{B} = B\boldsymbol{z}$ and $\boldsymbol{E} = E\boldsymbol{y}$. Here $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors.

   - (7 marks) Show that if the wave is consistent with Maxwell's laws, then $E = cB$. In addition, show that the wave moves with velocity $c = 1/\sqrt{\varepsilon\mu}$
   - (3 marks) Given that the energy densities associated with electric and magnetic fields are given by

     $$U_\varepsilon = \frac{\varepsilon}{2}E^2 \quad U_B = \frac{1}{2\mu}B^2$$

     Show that the power crossing unit area associated with a light wave is given by $S = \varepsilon c^2 EB$.

   Note: Faraday's and Ampere's law are given by:

   $$\oint E \cdot dl = -\frac{d\Phi}{dt} \quad \oint B \cdot dl = \mu \left[I + \varepsilon \frac{d(E \cdot A)}{dt} \right]$$

8. Scholarship Exam 2009 Paper 2 Q.4.

   - (5 marks) By considering the forces involved when light interacts with matter, and by deriving the real and imaginary parts of the dielectric function, show that in a transparent dielectric:

     $$n_1^2 = 1 + \frac{Ne^2}{\varepsilon_0 m} \frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

     Where all symbols have their usual meaning.

   - (3 marks) When light at optical frequencies falls on a material where the electrons are tightly bound, certain simplifications can be made to obtain an approximate version of the above equation. Demonstrate this to obtain a simplified equation relating n_1 to wavelength, λ.

   - Measurements have shown that the refractive index of crown grass varies with wavelength as shown below. Use the equation just derived to fit this data to obtain ω_0 for crown glass.

Wavelength (nm)	707	577	500	447	408
n_1, (m)	1.535	1.539	1.544	1.549	1.554

     Note: for most materials: $P = (\varepsilon_r - 1)\varepsilon_0 E$

9. Scholarship Exam 2008 Paper 2 Q.3.

   - (8 marks) Imagine a one dimensional light source of length D. Use the exponential representation for a spherical wave:

     $$E\frac{E_0}{r}e^{-i(\omega t - kr)}$$

     to show that the intensity of light falling on a far away screen is given by $I(\theta) = I(0)\sin c^2\beta$ where $\beta = (kD/2)y/L$ where y is the position on the screen and L is the distance from slit to screen. Indicate how this expression should be modified to represent a slit of width d illuminated from behind by plane polarised monochromatic light.

   - (1 mark) Write down an expression for the positions on the screen, y_{min} where the diffracted light is a minimum.

   - (1 mark) If the slit has width d = 1mm, the screen is L=5m away, and λ = 600nm, how wide is the central maximum?

   (Note: For a triangle of sides a, b, and c and opposing angles A, B, and C:

   $$a^2 = b^2 + c^2 - 2bc\cos A, \text{ furthermore, } (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots)$$

10. Scholarship Exam 2008 Paper 2 Q.4.

    - (3 marks) Explain what is meant by scattering of light. Use two of the laws of electricity and magnetism in your explanation.

    - (1 mark) Demonstrate why no light is scattered along the axis of the dipole.

- (2 marks) Using diagrams, explain the scattering of polarised light from a molecule. Use this to explain why sun-light is partially polarised.
- (2 marks) What is the difference between the directional distributions of light scattered from a single dipole versus light scattered from a regular array of dipoles.
- (2 marks) What is Brewster's angle? Calculate it for light incident from air on a smooth water surface. ($n_{\text{air}} = 0, n_{\text{water}} = 0.33$)

11. Scholarship Exam 2007 Paper 2 Q.3.
    - (Parts 1 and 2 same as 2009 Q.3. parts 1 and 2)
    - (1 mark) What is the Poynting vector?
    - (2 marks) For a harmonic electromagnetic wave, define the intensity, and show that it is given by $I = \varepsilon c E_0^2/2$ where E_0 is the amplitude of the electric field vector.

12. Scholarship Exam 2007 Paper 2 Q.4

    Consider two light waves travelling in the same direction

    $$\boldsymbol{E}_x = \boldsymbol{i}E_{0x}\cos(kz - \omega t) \qquad \boldsymbol{E}_y = \boldsymbol{i}E_{0y}\cos(kz - \omega t + \varepsilon)$$

    - (5 marks) Derive an equation for the projection onto the x-y plane of the curve traced by the tip of the resultant electric field vector.
    - (1 mark) What form does this curve take?
    - (2 marks) if the resultant light beam passes through a polariser whose transmission axis is in the x direction, what will the transmitted electric field and hence the transmitted intensity be?
    - (2 marks) What happens if the transmitted wave then passes through a quarter wave plate, whose optic axis is at $+45$ deg to the positive x axis. Describe the state of the Electric field vector with an equation.

13. Scholarship Exam 2006 Paper 2 Q.4.
    - (6 marks) By considering the forces acting on an electron in the presence of an oscillating E field, derive the real and imaginary parts of the relative permittivity of a dielectric.
    - (4 marks) How can these equations be modified to apply to a metal? use this to find an equation for the refractive index of a metal.

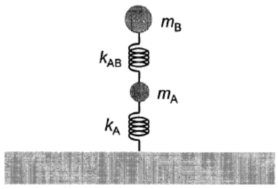
    Note: for dielectrics, the polarisation is related to the applied electric field by $P = (\varepsilon_r - 1)\varepsilon_0 E)$

3 Oscillations

1. TODO: 2012 Q.1
2. TODO: 2012 Q.2

3. Scholarship Exam 2011 Q.3.

   Nitric oxide (NO) bonds perpendicularly to the surface of platinum metal, with the nitrogen atom making the bond to the surface. The masses of the nitrogen, N, atom and oxygen, O, atoms are 14 and 16 atomic mass units, respectively. The interatomic force constant of the Pt-N bond is 600 Nm^{-1} and of the N-O bond is 1800 Nm^{-1}.



   - (5 marks) Show that the mode frequencies of the coupled oscillator system shown in the diagram above is given by

$$\omega^2 = \frac{1}{2}\left(\frac{k_A}{m_A} + \frac{k_{AB}}{m_B} + \frac{k_{AB}}{m_A}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k_A}{m_A} + \frac{k_{AB}}{m_B} + \frac{k_{AB}}{m_A}\right)^2 - \frac{k_A k_{AB}}{m_A m_B}}$$

   - (5 marks) Calculate the wavelengths at which infrared radiation is resonantly absorbed by the NO molecule when bonded to the surface.

4. Scholarship Exam 2011 Q.4.

   An elastic cord of length L is strung with N beads of mass m and seperation a.

   - (4 marks) If $-\omega^2 A_p = \omega_0^2 \{A_{p+1} - 2A_p + A_{p-1}\}$ and $w_0 = \sqrt{T_0/ma}$, where A_i is the displacement amplitude of the vibration of the ith bead at frequency ω, and T_0 is the tension in the cord, show that the nth mode frequency is given by

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

   - (2 marks) Re-express this equation in terms of the wavevector or wavenumber k_n of the nth mode.
   - (4 marks) A steel wire of length L has a dispersion relation $\omega = ck - dk^3$, deviating slightly from that of a perfectly elastic cord. Derive an expression, in terms of n and L, for the fractional change in the nth mode frequency, with respect to the same mode of a perfectly elastic cord.

5. Scholarship Exam 2010 Q.3.

   A damped harmonic oscillator of mass m, natural frequency ω_0, and width γ, is driven by the force $F = F_0 \cos \omega t$.

   - (2 marks) Derive, starting from the equations of motion, the displacement amplitude, $A(\omega)$, of the driven oscillator.
   - (2 marks) Use this expression to find the maximum value of $A(\omega)$.
   - (2 marks) Derive the value of the acceleration amplitude at $\omega = \omega_0$ and as $\omega \to \infty$.
   - (4 marks) Derive the resonance frequency of the acceleration amplitude.

6. Scholarship Exam 2010 Q.4. A vertical spring, of spring constant k, supports a mass, m. A second spring, of spring constant $2k$, connected to the first mass, and hanging vertically below it, supports a second mass, $2m$.

   - (3 marks) Determine the equations of motion of the two masses.
   - (3 marks) Find the frequencies of the normal modes of vibration of the system.
   - (4 marks) Both masses in the vertical arrangement are clamped in positions where the springs are unextended. If the clamp on the first mass is then released, the mass moves by 3cm. Determine the periods of the normal modes of vibration.

7. Scholarship Exam 2009 Paper 1 Q.3.

   The displacement amplitude, $A(\omega)$ of a damped harmonic oscillator of mass m, natural frequency ω_0, and width γ, and driven by a force $F = F_0 \cos \omega t$, is

$$A(\omega) = \frac{QF_0}{m} \frac{1}{\omega_0 \omega} \sqrt{\Re(\omega)}$$

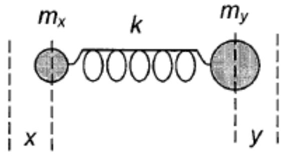
   where

$$\Re(\omega) \equiv \frac{(\gamma\omega)^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \text{ and } Q \equiv \frac{\omega_0}{\gamma}$$

   - (3 marks) Prove that $Q \approx A_{\omega_{\mathrm{wmax}}}/A_{\omega=0}$
   - (7 marks) Light of wavelength 600 nm is emitted by an electron in an atom behaving as a lightly damped single harmonic oscillator with a Q value of 5×10^7. Find the width, in nm, of the spectral line.

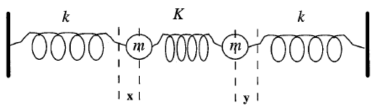
8. Scholarship Exam 2009 Paper 1 Q.4.

Two masses, m_x and m_y, are coupled together by a spring of stiffness, k, as shown in the figure.



- (2 marks) Derive the equations of motion of the two masses.
- (3 marks) Use these to show that $\omega^2 = \dfrac{k(m_x + m_y)}{m_x m_y}$
- (5 marks) If a carbon monoxide molecule, CO, has an interatomic force constant, k, of 2000 N m $^{-1}$ and the masses of the C and O atoms are 12 a.m.u and 16 a.m.u, respectively, find the wavelength of infrared radiation absorbed by the molecule.

9. Scholarship Exam 2008 Paper 1 Q.4.



The horizontal spring system shown above has general equations of motion of the form

$$\ddot{x} = -a_{11}x - a_{12}y$$
$$\ddot{y} = -a_{21}x - a_{22}y$$

which have solutions satisfying

$$(a_{11} - \omega^2)(a_{22} - \omega^2) - a_{21}a_{12} = 0$$

- (2 marks) Determine values for a_{ij} in terms of the masses, m, and the spring constant k and K.
- (3 marks) Find expressions for the two normal mode angular frequencies.
- (5 marks) When a single k spring is supported vertically and a mass, m, attached to its lower end, the mass oscillates vertically with a period of 1.2 s. When the K spring is treated in the same way the period is 1.0 s. Find the values of the normal mode angular frequencies of the system.

10. Scholarship Exam 2008 Paper 1 Q.4.

The nth mode frequency of an elastic cord strung with N beads of mass m and separation a is given by:

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

where $\omega_0 = \sqrt{\dfrac{T_0}{ma}}$ and T_0 is the tension in the cord.

- (2 marks) Re-express this equation in terms of the wavevector or wavenumber, k_n of the nth mode.
- (2 marks) Hence, or otherwise, show that, in the continuous limit, $\omega = v_\phi k$, where v_ϕ is the phase velocity of the wave.
- (4 marks) A piano string of length L has a dispersion relation given by $\omega = ck - dk^3$.

  Derive an expression for the fractional change in the nth mode frequency, with respect to the same mode of a perfectly flexible string.
- (2 marks) If $(d/c) = -1.0 \times 10^{-4}$ m^2 and $L = 1.0$m, find the fractional change, as defined above, in the frequency of the 10th mode, and state whether this change is an increase or decrease.

11. Scholarship Exam 2007 Paper 1 Q.3.

A light spring of spring constant, k, is suspended vertically from a fixed point. A load of mass m is attached to the lower end of the spring and the system attains equilibrium when the spring stretches by a length, h.

- (2 marks) Find the energy stored in the extended spring.
- (4 marks) When the load is displaced vertically through a small distance, a, and released from rest it oscillates around the equilibrium and eventually stops. Show that $b < 2\sqrt{km}$ where b is the resistance, and sketch the displacement of the load as a function of time. Discuss the main features of the graph.
- (4 marks) If the load has mass 0.2 kg, h is 0.05m, and b is 0.02kg s^{-1}, find

  (a) the period of oscillation

  (b) the time it takes for the amplitude to reduce by 80%

  (c) the number of complete oscillations the system performs in this time.

12. Scholarship Exam 2007 Paper 1 Q.4.

(Parts 1 and 2 same as 2008 Q.4 parts 1 and 2)

- (3 marks) Sketch the frequency behaviour of the beaded cord, and that of the simple, unbeaded cord, as a function of k, labelling your axes carefully. Discuss, in detail, the physics of the contrasting behaviour.
- (3 marks) The group velocity of a wave is defined a $v_g = (d\omega/dk)$. Show, firstly, that $v_g = v_\phi$ for a simple elastic cord. Show, secondly, that the *maximum* value of the phase velocity and the *minimum* value of the group velocity for the beaded cord occur at the same value of k.

13. Scholarship Exam 2006 Paper 1 Q.3.

The displacement amplitude, $A(\omega)$ of a damped harmonic oscillator of mass m, natural frequency ω_0, and width γ, and driven by a force $F = F_0 \cos \omega t$, is

$$A(\omega) = \frac{QF_0}{m} \frac{1}{\omega_0 \omega} \sqrt{\Re(\omega)}$$

where

$$\Re(\omega) \equiv \frac{(\gamma\omega)^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \text{ and } Q \equiv \frac{\omega_0}{\gamma}$$

- (2 marks) Prove that as $\omega \to 0$, $A(\omega) = \frac{F_0}{k}$, and as $w \to \infty$, $A(\omega) \to 0$.
- (2 marks) Use physical reasoning to explain why these results are expected.
- (2 marks) Prove that the acceleration amplitude is given by

$$A_{\text{acc}}(\omega) = \frac{QF_0}{m} \frac{\omega}{\omega_0} \sqrt{\Re(\omega)}$$

- (2 marks) Prove that as $\omega \to 0$, $A_{\text{acc}}(\omega) \to 0$, and, as $\omega \to \infty$, $A_{\text{acc}}(\omega) \to \frac{F_0}{m}$
- (2 marks) Use physical reasoning to explain why these results are expected.

14. Scholarship Exam 2006 Paper 1 Q.4.

(Parts 1 and 2 are the same as 2009 Q.4 parts 1 and 2)

- If a NaCl molecule has a natural vibrational frequency of 1.14×10^13 Hz, and the masses of the Na and Cl atoms are 23 a.m.u and 35 a.m.u., respectively, find the interatomic force constant of the molecule. (1 a.m.u $= 1.67 \times 10^{-27}$ kg)

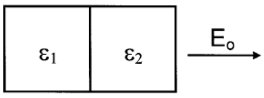
4 Electromagnetism

1. TODO: 2012 Q.1
2. TODO: 2012 Q.2

3. Scholarship Exam 2011 Q.9.

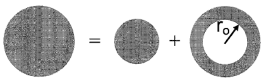
   Two dielectric cubes with dielectric constants ε_1 and ε_2 are bonded so that they share a common face. The cubes are placed in a uniform electric field, $\boldsymbol{E}_0$, in vacuum with the field parallel to the long axis of the bonded cubes, as shown.

   - (4 marks) Derive expressions for the displacement *and* polarisation fields in each dielectric cube in terms of ε_0, ε_1, ε_2, and $\boldsymbol{E}_0$.
   - (3 marks) Derive expressions for the surface charge densities which are induced by the field $\boldsymbol{E}_0$ at the left and right faces of the bonded dielectric cubes and at the interface between the cubes.
   - (3 marks) A free charge density, σ_{free} is added at the interface between the two cubes. Derive new expressions for the displacement *and* polarisation fields in each dielectric cube when this charge is present.



4. Scholarship Exam 2011 Q.10.

   The electrostatic potential at a distance r_0 from the centre of a spherical symmetric charge distribution $\rho(r)$ can be calculated by applying Gauss' Law and using the spherical symmetry of the distribution. The charge is divided into *filled* and *hollow* spheres of radius r_0 as shown. The charge in the filled sphere is $Q(r_0)_{\text{encl}}$.



   - (2 marks) Apply Gauss' law in its integral form to the filled sphere to show that the magnitude of the electric field, E, at the surface of that sphere is
   $$E = \frac{Q(r_0)_{\text{encl}}}{4\pi\varepsilon_0 r_0^2}$$
   - (3 marks) Apply Gauss' law in its differential form to the hollow sphere to show that the electric field is zero along a line from the inner surface of the hollow sphere at r_0 to the empty centre of the sphere. *Hint: Consider the symmetric properties of the electric field along this line.*
   - (3 marks)Show that the potential at the inner surface of the hollow sphere, due to the hollow sphere charge only, is given by:
   $$\varphi(0) = \varphi(r_0) = \frac{1}{\varepsilon_0}\int_{r_0}^{\infty} \rho(r) r dr$$

- (2 marks) Hence show that the total potential at r_0 is given by:

$$\varphi(r_0) = \frac{1}{\varepsilon_0} \int_0^{r_0} \rho(r) r dr + \frac{1}{\varepsilon_0 r_0} \int_{r_0}^{\infty} \rho(r) r^2 dr$$