We use the fact that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}$

let $A = \vartheta/2$; $c = \cos$; $s = \sin$

We have $e^{-i\sigma_y A} = c(\sigma_y A) + i s(\sigma_y A)$ (1)

Taylor expansions for cos and sine:

$$c(X) = \mathbb{1} - \frac{X^2}{2!} + \frac{X^4}{4!} - \dots$$

$$s(X) = X - \frac{X^3}{3!} + \frac{X^5}{5!} - \dots$$

so $c(\sigma_y A) = \mathbb{1} - \sigma_y^2 \frac{A^2}{2!} + \sigma_y^4 \frac{A^4}{4!} - \dots$

$$= \mathbb{1} - \mathbb{1}\frac{A^2}{2!} + \mathbb{1}\frac{A^4}{4!} - \dots$$

$$= \mathbb{1}\left(1 - A^2/2! + A^4/4!\right) = \mathbb{1}\, c(A) = \mathbb{1}\cos\left(\vartheta/2\right)$$

and $s(\sigma_y A) = \sigma_y A - \sigma_y^3 \frac{A^3}{3!} + \sigma_y^5 \frac{A^5}{5!} - \dots$

$$= \sigma_y\left[A - \sigma_y^2 \frac{A^3}{3!} + \sigma_y^4 \frac{A^5}{5!} - \dots\right]$$

$$= \sigma_y\left[A - \mathbb{1}A^3/3! + \mathbb{1}A^5/5! - \dots\right]$$

$$= \sigma_y\left[A - A^3/3! + A^5/5!\right]$$

$$= \sigma_y s(A) = \sigma_y \sin\left(\vartheta/2\right)$$

$\Rightarrow$ (1) $= \mathbb{1}\cos\left(\vartheta/2\right) + i\sigma_y \sin\left(\vartheta/2\right)$ as required.