Problem set 1

Pr. 1. $L = \frac{1}{2} m v_i^2 - \frac{\varkappa}{r} = \frac{1}{2} m \dot{x}_i^2 - \frac{\varkappa}{r}$

$$\delta L = -m \ddot{x}_i \delta x_i + \frac{\varkappa}{r^2} \delta r = -m \ddot{x}_i \delta x_i + \frac{\varkappa}{r^3} x_i \delta x_i = 0$$

$$\boxed{m \ddot{x}_i = \varkappa \frac{x_i}{r^3}}$$

$$x_i \to O_{ij} x_j \Rightarrow v_i \to O_{ij} v_j$$

$L \to L$ because

$$x_i^2 \to O_{ij} x_j O_{ik} x_k = O_{ij} O_{ik} x_j x_k = \delta_{jk} x_j x_k = x_i^2$$

Pr. 2. $L = \frac{1}{2} m_{ij} v_i v_j - \frac{1}{2} k_{ij} x_i x_j$

$$\delta L = -\frac{1}{2}(m_{ij} + m_{ji}) \ddot{x}_j \delta x_i - \frac{1}{2}(k_{ij} + k_{ji}) x_j \delta x_i = 0$$

$$\boxed{\frac{1}{2}(m_{ij} + m_{ji}) \ddot{x}_j + \frac{1}{2}(k_{ij} + k_{ji}) x_j = 0}$$

$D = 1$; $m\ddot{x} + kx = 0 \Rightarrow x = a \cos\left(\sqrt{\frac{k}{m}}\, t + \varphi\right)$

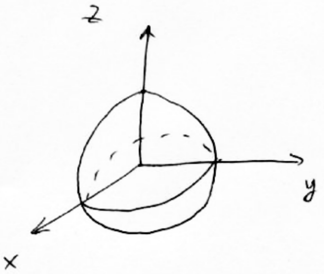
Pr. 3. $L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2} \dot{x}_i^2}$

$$\delta L = m_0 c^2 \frac{\frac{1}{c^2} \dot{x}_i \delta \dot{x}_i}{\sqrt{1 - \frac{1}{c^2} \dot{x}_j^2}} = -m_0 \left(\frac{\dot{x}_i}{\sqrt{1 - \frac{1}{c^2} \dot{x}_j^2}} \right)^{\bullet} \delta x_i = 0$$

$$\boxed{\left(\frac{\dot{x}_i}{\sqrt{1 - \frac{1}{c^2} \dot{x}_j^2}} \right)^{\bullet} = 0} \Rightarrow \frac{v_i}{\sqrt{1 - \frac{1}{c^2} v^2}} = \text{const} \Rightarrow$$

$$\Rightarrow v_i = \text{const} \quad ; \quad v^2 < c^2$$

Pr. 4



$$x^2+y^2+z^2=R^2$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - mgz$$

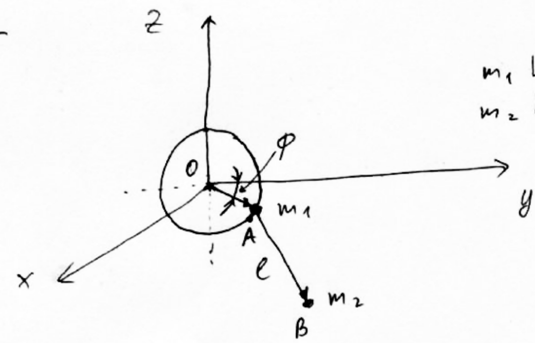
Constraint: $x^2+y^2+z^2=R^2$

spherical coordinates: $z = R\cos\theta$; $x = R\sin\theta\cos\varphi$; $y = R\sin\theta\sin\varphi$

$$L = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\varphi}^2) - mgR\cos\theta = |\dot{R}=0| \Rightarrow$$

$$\boxed{L = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) - mgR\cos\theta}$$

Pr. 5



$x \perp$ the circle

m_1 has coordinates (x_1, y_1, z_2)

m_2 has coordinates (x_2, y_2, z_2)

Constraints:

$C_1 = x_1 = 0$ (circle lies in the yz-plane)

$C_2 = y_1^2 + z_1^2 - R^2 = 0$ (m_1 is on the circle)

$C_3 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - \ell^2 = 0$ (distance between m_1 and m_2 is ℓ).

Problem set 1

Pr. 5 (continued)

Independent coordinates:

1) ϕ is the angle between $\vec{OA}$ and the y-axis
2) θ is the angle between $\vec{AB}$ and the z-axis
3) φ is the angle between the x-axis and the projection of $\vec{AB}$ onto the xy-plane

θ and φ are the angles of the spherical coordinates

Solving the constraints

$$x_1 = 0 \,; \quad y_1 = R\cos\phi \,; \quad z_1 = R\sin\phi \qquad (\phi \text{ can be negative})$$

$$x_2 = x_1 + \ell \sin\theta\cos\varphi = \ell\sin\theta\cos\varphi$$

$$y_2 = y_1 + \ell\sin\theta\sin\varphi = R\cos\phi + \ell\sin\theta\sin\varphi$$

$$z_2 = z_1 + \ell\cos\theta = R\sin\phi + \ell\cos\theta$$

Substituting into

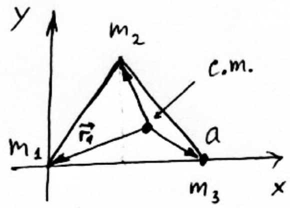
$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - m_1 g z_1 - m_2 g z_2$$

we get

$$\boxed{\begin{aligned} L = {} & \frac{R^2}{2}(m_1+m_2)\dot{\phi}^2 - m_2 \ell R \dot{\phi}\Big((\cos\phi\sin\theta + \cos\theta\sin\phi\sin\varphi)\dot{\theta} + \\ & + \cos\varphi\sin\phi\sin\theta\,\dot{\varphi}\Big) + \frac{1}{2} m_2 \ell^2 \left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - \\ & - g(m_1+m_2)R\sin\phi - g m_2 \ell\cos\theta \end{aligned}}$$

Problem set 2

Pr. 1



$x_1 = y_1 = 0$

$x_2 = \frac{a}{2}$; $y_2 = \frac{\sqrt{3}}{2} a$

$x_3 = a$; $y_2 = 0$ at $t=0$

$m_1 = m$; $m_2 = 2m$; $m_3 = 3m$; $\mu = \sum_a m_a = 6m$

$$X_{cm} = \frac{x_a m_a}{\mu} = \frac{2}{3} a ;$$

$$y_{cm} = \frac{y_a m_a}{\mu} = \frac{\sqrt{3}}{6} a ;$$

a) $\vec{f}_a = \dot{\vec{p}}_a = m_a \dot{\vec{V}}_a$

$\vec{V}_a = \vec{\omega} \times \vec{r}_a$; $\vec{r}_a$ is the vector from c.m. to m_a.

$\dot{\vec{V}}_a = \vec{\omega} \times \dot{\vec{r}}_a = \vec{\omega} \times \vec{V}_a = \vec{\omega} \times (\vec{\omega} \times \vec{r}_a) = -\omega^2 \vec{r}_a$

Thus, $\vec{f}_a = -\omega^2 m_a \vec{r}_a$, no summation over a.

Computing $\vec{r}_a$ at $t=0$, we get

$$\vec{f}_1 = \left(\frac{2}{3} , \frac{\sqrt{3}}{6} \right) m\omega^2 a$$

$$\vec{f}_2 = \left(\frac{1}{3} , - \frac{2\sqrt{3}}{3} \right) m\omega^2 a$$

$$\vec{f}_3 = \left(-1 , \frac{\sqrt{3}}{2} \right) m\omega^2 a$$

$$\vec{f}_1 + \vec{f}_2 + \vec{f}_3 = 0$$

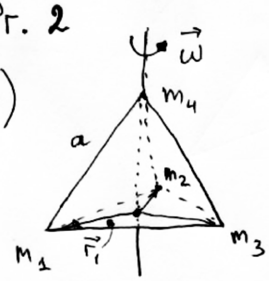
b) $\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a \vec{r}_a \times m_a (\vec{\omega} \times \vec{r}_a) = \vec{\omega} \sum_a m_a \vec{r}_a^{\,2} \Rightarrow$

$$\vec{M} = 2ma^2 \vec{\omega}$$

Problem set 2

Pr. 2

i)



a) $\vec{f}_4 = 0$; at $t=0$

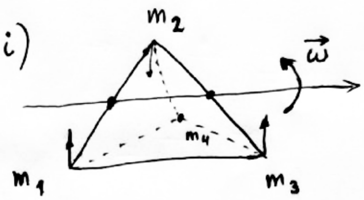
$\vec{f}_i = -\omega^2 m \vec{r}_i$; $i = 1, 2, 3.$

(see Pr. 1)

$|\vec{f}_i| = \frac{1}{\sqrt{3}} m\omega^2 a$;

b) $\vec{M} = \vec{\omega} \sum_a m_a \vec{r}_a^{\,2} = ma^2 \vec{\omega}$;

ii)



a) $\vec{f}_a = -\omega^2 m \vec{r}_a$

$\vec{r}_a$ is the vector from the axes of rotation to m_a, and $\perp$ to $\vec{\omega}$.

$|\vec{f}_i| = \frac{\sqrt{3}}{4} \omega^2 m a$, $i = 1, 2, 3$

$|\vec{f}_4| = \sqrt{\frac{11}{16}}\, \omega^2 m a$

b) $\vec{M} = \vec{\omega} \cdot \sum m_a r_a^2 = \frac{5}{4} m a^2 \vec{\omega}$

Pr. 3 : $L = \frac{1}{2} m \vec{v}^{\,2} - \frac{\varkappa}{r}$

$x_i \to x_i + \varepsilon_{ij} x_j \Rightarrow \delta x_i = \varepsilon_{ij} x_j$

$\frac{1}{2} \mathcal{J}_{ij} \cdot \varepsilon_{ij} = \frac{\partial L}{\partial v_i} \cdot \delta x_i = m v_i \varepsilon_{ij} x_j = \frac{1}{2} (p_i x_j - p_j x_i) \varepsilon_{ij}$

Thus, $\boxed{\mathcal{J}_{ij} = p_i x_j - p_j x_i}$ is conserved

$\mathcal{J}_{ij}$ is skew-symmetric : $\mathcal{J}_{ij} = -\mathcal{J}_{ji}$, therefore

$\exists\ \frac{D(D-1)}{2}$ independent charges.
