Problem set 1

Pr. 1. $L = \frac{1}{2} m v_i^2 - \frac{\varkappa}{r} = \frac{1}{2} m \dot{x}_i^2 - \frac{\varkappa}{r}$

$$\delta L = -m\ddot{x}_i \delta x_i + \frac{\varkappa}{r^2}\delta r = -m\ddot{x}_i \delta x_i + \frac{\varkappa}{r^3} x_i \delta x_i = 0$$

$$\boxed{m\ddot{x}_i = \varkappa \frac{x_i}{r^3}}$$

$$x_i \to O_{ij} x_j \Rightarrow v_i \to O_{ij} v_j$$

$L \to L$ because

$$x_i^2 \to O_{ij} x_j O_{ik} x_k = O_{ij} O_{ik} x_j x_k = \delta_{jk} x_j x_k = x_i^2$$

Pr. 2. $L = \frac{1}{2} m_{ij} v_i v_j - \frac{1}{2} k_{ij} x_i x_j$

$$\delta L = -\frac{1}{2}(m_{ij} + m_{ji})\ddot{x}_j \delta x_i - \frac{1}{2}(k_{ij} + k_{ji}) x_j \delta x_i = 0$$

$$\boxed{\frac{1}{2}(m_{ij} + m_{ji})\ddot{x}_j + \frac{1}{2}(k_{ij} + k_{ji}) x_j = 0}$$

$D = 1$; $m\ddot{x} + kx = 0 \Rightarrow x = a\cos\left(\sqrt{\frac{k}{m}}\, t + \varphi\right)$

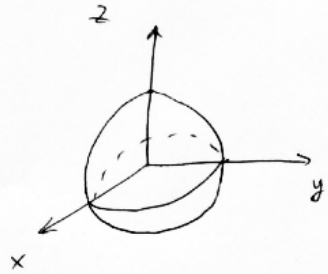
Pr. 3. $L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2}\dot{x}_i^2}$

$$\delta L = m_0 c^2 \frac{\frac{1}{c^2}\dot{x}_i \delta \dot{x}_i}{\sqrt{1 - \frac{1}{c^2}\dot{x}_j^2}} = -m_0 \left(\frac{\dot{x}_i}{\sqrt{1 - \frac{1}{c^2}\dot{x}_j^2}}\right)^{\bullet} \delta x_i = 0$$

$$\boxed{\left(\frac{\dot{x}_i}{\sqrt{1 - \frac{1}{c^2}\dot{x}_j^2}}\right)^{\bullet} = 0} \Rightarrow \frac{v_i}{\sqrt{1 - \frac{1}{c^2}v^2}} = \text{const} \Rightarrow$$

$$\Rightarrow v_i = \text{const} \;;\quad v^2 < c^2$$

Pr. 4



$$x^2+y^2+z^2=R^2$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - mgz$$

Constraint: $x^2+y^2+z^2=R^2$

spherical coordinates: $z = R\cos\theta$; $x = R\sin\theta\cos\varphi$; $y = R\sin\theta\sin\varphi$

$$L = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\varphi}^2\right) - mgR\cos\theta = \left|\dot{R}=0\right| \Rightarrow$$

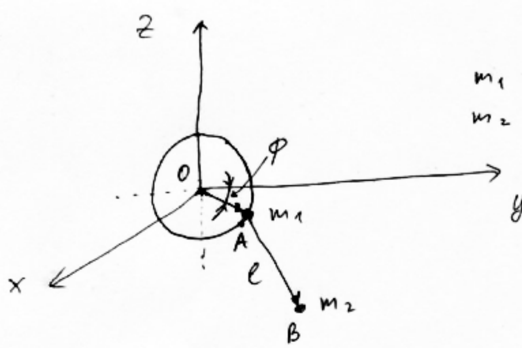
$$\boxed{L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - mgR\cos\theta}$$

Pr. 5

$x \perp$ the circle

m_1 has coordinates (x_1, y_1, z_2)

m_2 has coordinates (x_2, y_2, z_2)



Constraints:

$c_1 = x_1 = 0$ (circle lies in the yz-plane)

$c_2 = y_1^2 + z_1^2 - R^2 = 0$ (m_1 is on the circle)

$c_3 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - \ell^2 = 0$ (distance between m_1 and m_2 is ℓ).

Problem set 1 ③

Pr. 5 (continued)

Independent coordinates:

1) ϕ is the angle between $\overrightarrow{OA}$ and the y-axis
2) θ is the angle between $\overrightarrow{AB}$ and the z-axis
3) φ is the angle between the x-axis and the projection of $\overrightarrow{AB}$ onto the xy-plane

θ and φ are the angles of the spherical coordinates

Solving the constraints

$$x_1 = 0\,; \quad y_1 = R\cos\phi\,; \quad z_1 = R\sin\phi \qquad (\phi \text{ can be negative})$$

$$x_2 = x_1 + \ell\sin\theta\cos\varphi = \ell\sin\theta\cos\varphi$$

$$y_2 = y_1 + \ell\sin\theta\sin\varphi = R\cos\phi + \ell\sin\theta\sin\varphi$$

$$z_2 = z_1 + \ell\cos\theta = R\sin\phi + \ell\cos\theta$$

Substituting into

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - m_1 g z_1 - m_2 g z_2$$

we get

$$\boxed{\begin{aligned} L = {} & \frac{R^2}{2}(m_1+m_2)\dot{\phi}^2 - m_2 \ell R\,\dot{\phi}\Big(\big(\cos\phi\sin\theta + \cos\theta\sin\phi\sin\varphi\big)\dot{\theta} + \\ & + \cos\varphi\sin\phi\sin\theta\,\dot{\varphi}\Big) + \frac{1}{2}m_2\ell^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - \\ & - g(m_1+m_2)R\sin\phi - g m_2 \ell\cos\theta \end{aligned}}$$