Advanced Classical Mechanics

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1 Introduction

This is essentially a LaTeX version of course notes taken from Sergei Frolov's lectures in Classical Mechanics (MA2341) given during Michaelmas term 2011. While starting as a close to verbatim transcription of what appeared on the blackboard, the intention is that which each successive revision the file here should steadily diverge as the flow and presentation of ideas are improved and expository notes are added. Any errors noted or suggestions should be emailed to the author at allenam@tcd.ie. The notes do not cover the second semester for the reason that my handwritten notes, in contrast to those for the first semester, are adequate. However, I may scan the second semester notes if I feel the need or there is demand.

1.1 Additional Material

In addition to Landau & Lifshitz, two other resources are especially useful for understand Lagrangian and Hamiltonian formulations, the first is Chapter 19 of Volume II of Feynman's *Lectures on Physics*, which gives a very nice, conversational and intuitive introduction to the principle of least action. The second is Cornelius Lanczos' *Variational Principles of Mechanics* which, while aimed at a graduate audience, takes an approach which is rather philosophical and light on maths. Lanczos is especially useful in placing the principle of least action on firm theoretical ground and traces its roots from the principle of virtual work and d'Alembert's principle.

2 The Principle of Least Action

2.1 Generalised Coordinates

2.2 Cartesian Coordinates

Consider a particle in space $\mathbf{r} = (x, y, z)$ We wish to know $\mathbf{r}(t) \forall t$. i.e. the position of the particle at all times t.

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta \mathbf{t}} = \frac{dr}{dt} = \dot{r}$$

$$\mathbf{v} = (v_x, v_y, v_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

$$|v| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} = \frac{ds}{dt} = \text{speed}$$

s is the length of the path of the particle

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij} dx^i dx^j \text{ (summation over repeated indices)}$$

$$x^1 = x,\ x^2 = y,\ x^3 = z$$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

2.2.1 Einstein notation

Einstein notation is a notation that makes it easier to write formulae by making summations implicit: Whenever an index variable appears twice in a single term it implies summation of that term. So for instance, if we have $g_{ij} dx^i dx^j$ then i and j are repeated, and thus the formula is short hand for the summation

$$\sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} dx^i dx^j$$

$$= \sum_{i=1}^{3} \left[g_{i1} dx^i dx^1 + g_{i2} dx^i dx^2 + g_{i3} dx^i dx^3 \right]$$

$$= g_{11} dx^1 dx^1 + \cancel{g_{21} dx^2 dx^1} + \ldots$$

$$= (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

g_{ij} is the metric of $\mathbb{R}^3$ in cartesian coordinates, $ds^2 = g_{ij} dx^i dx^j$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$$

This is the cartesian coordinate system, however there are different coordinate systems we could use:

2.3 Polar/Cylindrical coordinates

r is the projection of ρ onto the xy-plane $\rho = (x, y, z) \to (r, \phi, z)$

$$\begin{aligned} r &= R \leftarrow \text{cylinder of radius } R \\ z &= c \leftarrow \text{plane parallel to } xy \\ \phi &= c \leftarrow \text{ semiplane perpendicular to } xy \end{aligned}$$

2.3.1 First derivatives:

We have $\dot{r}, \dot{z}, \dot{\phi}$ (ϕ is angular velocity).

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \text{ (cartesian)} \\ &= dr^2 + r^2 d\phi^2 + dz^2 \text{ (polar/cylindrical)} \\ &= g_{ij} dx^i dx^j \end{aligned}$$

$$x^1 = r,\ x^2 = \phi,\ x^3 = z$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.4 Spherical Coordinates

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

$$\begin{aligned} r &= R \text{ is sphere of radius } R \\ \phi &= \text{const is a semiplane} \perp xy \\ \theta &= \text{const is cone} \end{aligned}$$

$$ds^2 = dr^2 + r^2 d\theta + r^2 \sin^2\theta d\theta^2 + g_{ij} dx^i dx^j$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

2.5 Degrees of Freedom

If we have N particles in space, we need $3N$ coordinates (in any coordintes).

The number of independent quantities which must be specified to define uniquely the position of any system is called the number of *degrees of freedom.*

Consider a hyperboloid ($-x^2 + y^2 + z^2 = -R2$)

However, if two are connected we have $6 - 1 = 5$.

Figure 1: $3 * 2 = 6$ degrees of freedom

Figure 2: $(4 * 3) - 6 = 6$ degrees of freedom

The degrees of freedom is given by the number of particles times the number of dimensions. Each constraint reduces the number of degrees of freedom by one.

e.g. the 4 particle system in 3-d space:

However, we only consider *independent constraints*.

Any s quantities $q_1, \ldots, q_s$ which define the position of a system with s degrees of freedom are called *generalised coordinates*.

$$v_i = \dot{q}_i = \text{ generalised velocity}$$

e.g. $\dot{\phi}$ = generalised velocity of ϕ = angular velocity of rotation. For some systems it is not sufficient to know just velocities and positions, we must know acceleration too. However we are not concerned with those. So, we have systems where we can compute

$$\begin{aligned} q_i(t_0) &\to q_i(t) \forall t \\ v_i(t_0) = \dot{q}_i(t_0) &\nearrow \end{aligned}$$

From this, we must have

$$\left.\begin{array}{ll} \ddot{q}_i & = f_i(q_j, \dot{q}_j) \\ & \text{or} \\ \ddot{q} & = f(q, \dot{q}) \end{array}\right\} \text{equations of motion of the system}$$

Why can we do this? Consider:

$$\begin{aligned} \dddot{q}_i &= \frac{d}{dt} f_i(q_j, \dot{q}_j) \\ &= \underbrace{\frac{\partial f_i}{\partial q_i}\dot{q}_j + \underbrace{\frac{\partial f_i}{\partial q_i}\ddot{q}_j}_{=f(q,\dot{q})}}_{=F(q,\dot{q})} \end{aligned}$$

and we can continue this principle finding derivatives further down.

2.6 Hamilton's Principle of Least Action

1. Any mechanical system is characterised by a definite function (called the Lagrangian)

   $$L(q_1, \dots, q_s, \dot{q}_1, \dots, \dot{q}_s, t)$$

   Introduce

   $$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

   which is the action of the system.

2. Let the system have coordinates $q_i^{(1)}$ and $q_i^{(2)}$ at t_1 and t_2. Then the system moves between $q_i^{(1)}$ and $q_i^{(2)}$ in such a way that the action S takes the least possible value (in general an extremum).

Let $q(t)$ be the function for which S is minimised, so $q(t)$ is the actual path of the system. Replace $q(t)$ with $q(t) + \delta q(t)$ where $\delta q(t)$ is 'small' everywhere on $[t_1, t_2]$ ($\delta q(t)$ is a *variation* of $q(t)$)

Figure 3: $\delta q(t_1) = \delta q(t_2) = 0$

$$\begin{aligned}
\Delta S &= \Delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \ \leftarrow \text{difference in two actions } q \text{ and } q + \delta q \\
&= \int_{t_1}^{t_2} dt \left(L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t) \right) \\
&\approxeq \underbrace{\int_{t_1}^{t_2} dt \, F(q, \dot{q}, t) \delta q}_{\text{which must vanish}} + \int_{t_1}^{t_2} dt \, G(q, \dot{q}, t) (\delta q)^2 + \dots
\end{aligned}$$

2.6.1 Example: Deriving equation of motion for a given Lagrangian

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 = mv^2$$

$$S = \int_{t_1}^{t_2} L\,dt = \frac{m}{2}\int_{t_1}^{t_2} \dot{\mathbf{r}}^2 dt$$

$$\Delta S = \frac{m}{2}\int_{t_1}^{t_2} \left[(\dot{\boldsymbol{r}} + \delta\dot{\boldsymbol{r}})^2 - \dot{\boldsymbol{r}}^2\right] dt = \int_{t_1}^{t_2} m\dot{\boldsymbol{r}}\delta\dot{\boldsymbol{r}} + \underbrace{\frac{m}{2}(\delta\dot{\boldsymbol{r}})^2}_{\text{We discard this term (negligible)}} dt$$

$$= \underbrace{m\dot{\boldsymbol{r}}\delta\mathbf{r}\Big|_{t_1}^{t_2}}_{0 \text{ since } \delta r(t_1) = \delta r(t_2)} + \int_{t_1}^{t_2} (-m\ddot{\boldsymbol{r}})\,\delta r\,dt$$

So:

$$\left.\delta S = \int_{t_1}^{t_2} (-m\ddot{\boldsymbol{r}}\delta r)\,dt = 0 \quad\right\}\text{ first variation of } S$$

$$\implies m\ddot{\boldsymbol{r}} = 0 \implies \mathbf{v} = \text{const}$$

Note that the above uses integration by parts, see the footnote[1]for explanation if this is confusing.

2.6.2 Another example:

$$L = \frac{m}{2}\ddot{\boldsymbol{r}}^2 + \mathbf{F}(t)\cdot\mathbf{r}\ , \quad (\mathbf{F}(t) \text{ is a vector independent of } \mathbf{r}, \dot{\boldsymbol{r}})$$

So:

$$\delta S = \int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} dt\,[-m\ddot{\boldsymbol{r}} + \mathbf{F}(t)]\cdot\delta\mathbf{r} = 0$$

$$\implies m\ddot{\boldsymbol{r}} = \mathbf{F}(t)$$

which is Newton's 2nd law for a particle acted upon by an external field.

---

[1] Note that $\frac{d}{dt}m\dot{r}\delta r = m\ddot{r}\delta r + m\dot{r}\delta\dot{r}$

So (rearranging and integrating) $\int_{t_1}^{t_2} (m\dot{r}\delta\dot{r})\,dt = \Big[m\dot{r}\delta r\Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{r}\delta r\,dt$

2.7 Lagrangian of multiple particles:

$$L = \underbrace{\sum_{a=1}^{N} \frac{m_a}{2} \dot{\boldsymbol{r}}_a^2}_{\text{KE}} - \underbrace{U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, t)}_{\text{PE}}$$

We wish, as usual, to find the variation of δS:

$$\delta S = \int_{t_1}^{t_2} \sum_{a=1}^{N} (-m_a \ddot{\boldsymbol{r}}_a \delta \boldsymbol{r}_a)\, dt - \int_{t_1}^{t_2} \delta U \, dt$$

So[2]: $\delta S = \int_{t_1}^{t_2} \left[\sum_{a=1}^{N} \left(-m_a \ddot{\boldsymbol{r}}_a - \frac{\partial U}{\partial \boldsymbol{r}_a} \right) \cdot \delta \boldsymbol{r}_a \right] dt$

2.8 The general Lagrangian

Now we derive general equations for the general Lagrangian.

$$S = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\, dt; \qquad \boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_s)$$

$$\dot{\boldsymbol{q}} = (\dot{\boldsymbol{q}}_1, \dot{\boldsymbol{q}}_2, \ldots, \dot{\boldsymbol{q}}_s)$$

---

[2] This might be confusing if you do not understand partial differentiation properly. This footnote should explain the steps involved:

$$\delta U = U(\boldsymbol{r}_1 + \delta \boldsymbol{r}_1,\, \boldsymbol{r}_2 + \delta \boldsymbol{r}_2,\, \ldots,\, \boldsymbol{r}_n + \delta \boldsymbol{r}_n,\, t) - U(\boldsymbol{r}_1,\, \boldsymbol{r}_2,\, \ldots,\, \boldsymbol{r}_{\boldsymbol{n}}, t)$$

The Taylor Series gives us:

$$\begin{aligned} f(x + \delta x) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(\delta x)^n \\ &\approxeq f(x) + f'(x)\delta x \end{aligned}$$

so:

$$\delta f \approxeq f(x + \delta x) - f(x) = f'(x)\delta x$$

so:

$$\begin{aligned} f(x_1 + \delta x_1, \ldots, x_n + \delta x_n) &= \sum_{n_1=0}^{\infty} \frac{1}{n_1!} f^{(n_1, 0, \ldots, 0)}(x_1, x_2 + \delta x_2, \ldots, x_n + \delta x_n) \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{1}{n_1!} \frac{1}{n_2!} f^{(n_1, n_2, 0, \ldots, 0)}(x_1, x_2, x_3 + \delta x_3, \ldots) \\ &= \sum_{n_1, n_2, \ldots, n_N = 0}^{\infty} \frac{f^{(n_1, n_2, \ldots, n_N)}}{n_1!, n_2!, \ldots, n_N!} \boldsymbol{x} \delta x_1, \delta x_2, \ldots, \delta x_N \end{aligned}$$

so:

$$\begin{aligned} \delta U &= \frac{\partial U}{\partial \boldsymbol{r}_1} \cdot \delta \boldsymbol{r}_1 + \frac{\partial U}{\partial \boldsymbol{r}_2} \cdot \delta \boldsymbol{r}_2 + \ldots + \frac{\partial U}{\partial \boldsymbol{r}_n} \cdot \delta \boldsymbol{r}_n \\ &= \sum_{n=1}^{N} \frac{\partial U}{\partial \boldsymbol{r}_n} \delta \boldsymbol{r}_n \end{aligned}$$

By now it should be clear from the preceding that this is a straightforward exercise.[3]

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt\,[L(\boldsymbol{q}+\delta\boldsymbol{q},\dot{\boldsymbol{q}}+\delta\dot{\boldsymbol{q}},t)-L(\boldsymbol{q},\dot{\boldsymbol{q}},t)] \\
&= \int_{t_1}^{t_2} dt\left[\frac{\partial L}{\partial \boldsymbol{q}}\cdot\delta\boldsymbol{q}+\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\cdot\delta\boldsymbol{q}\right)\right] \\
&= \underbrace{\left.\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\cdot\delta\boldsymbol{q}\right|_{t_1}^{t_2}}_{0\text{ since }\delta\boldsymbol{q}(t_1)=\delta\boldsymbol{q}(t_2)=0} + \int_{t_1}^{t_2} dt\left(\frac{\partial L}{\partial \boldsymbol{q}}-\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\cdot\delta\boldsymbol{q} \\
&\Longrightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}=\frac{\partial L}{\partial \boldsymbol{q}}}
\end{aligned}
$$

which is Lagrange's equation of motion. Note that $\frac{\partial L}{\partial \boldsymbol{q}}=\left(\frac{\partial L}{\partial \boldsymbol{q}_1},\ldots,\frac{\partial L}{\partial \boldsymbol{q}_s}\right)$ and $\frac{\partial L}{\partial \dot{\boldsymbol{q}}}=\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1},\ldots,\frac{\partial L}{\partial \dot{\boldsymbol{q}}_s}\right)$ so we take components and just equate them. So explicitly,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}^i}=\frac{\partial L}{\partial \boldsymbol{q}^i}; \qquad\qquad i=1,\ldots,s$$

This is a 2nd order differential equation (i.e. $\frac{d}{dt}f(\dot{\boldsymbol{q}})=f(\ddot{\boldsymbol{q}})$), so in general to solve we need coordinates $q^i(t_0)$ and velocities $\dot{q}^i(t_0)$

2.9 Various properties of the Lagrangian

Additivity

$$\lim_{|AB|\to\infty} L=(L_A+L_B+L_{int})\to L_A+L_B$$

i.e. for two subsystems A and B, described by Langrangians L_A and L_B respectively, with an interaction Lagrangian L_{int}, the Lagrangian for the full system is given by L_A+L_B as the distance between them tends towards infinity i.e. as the interaction tends towards zero.

Multiplicity

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}=\frac{\partial L}{\partial q}\leftarrow \text{ linear in } L$$

$$L\to gL; \qquad\qquad g\text{ is const}$$

e.g. $L=\frac{m}{2}v^2$ $m\to gm$ i.e. we can just multiply by a scale constant.

---

[3] This uses integration by parts, look at previous footnote and note that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)=\frac{\partial L}{\partial \dot{q}}\delta\dot{q}+\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

and it should be clear. The reason why we transform the equation in this way is that we can remove all instances of $\delta\dot{q}$ and only have δq in the final integral.

Total derivative freedom

$$\tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

Equations of motion are the same for $\tilde{L}$ and L since:

$$\tilde{S} = \int_{t_1}^{t_2} \tilde{L}\, dt = S + \int_{t_1}^{t_2} \frac{d}{dt} f\, dt = S + f(q(t_2), t_2) - f(q(t_1), t_1)$$

So $\delta \tilde{S} = \delta S$ since $\delta q(t_1) = \delta q(t_2) = 0$ This means that two Lagrangians give rise to identical equations of motion if they differ only by some total time derivative. This turns out to be very important and useful.

2.9.1 Example

$$\begin{aligned} &L = T - U \\ &\tilde{L} = L - \frac{d}{dt}(ct) \\ &U \to U + c \end{aligned}$$

i.e. we can add any constants to potential energy.

2.10 Systems of Particles

$$L = \sum_{a=1}^{N} \frac{m_a}{2} \boldsymbol{v}_a^2 - U(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n)$$

We look at the properties of this Lagrangian.

If there is no explicit time dependence, then the system is closed.

1. U depends only on $\boldsymbol{r}_a$

   This implies that interactions propagate instantaneously (related to the *absolute nature of time*, the main assumption here)

2. $t \to t + a$ (time is homogeneous)
3. $t \to -t$ (time is isotropic)

   (Reflection property or $\mathbb{Z}_2$ symmetry of L)

Equation of motion:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}_a} &= \frac{\partial L}{\partial \boldsymbol{r}_a} \\ \implies m_a \dot{\boldsymbol{v}}_a &= -\frac{\partial U}{\partial \boldsymbol{r}_a} \equiv \boldsymbol{F}_a \end{aligned}$$

So:

$$\boldsymbol{F} = \text{const} \leftarrow \text{ Uniform field force}$$
$$U = -\boldsymbol{F} \sum_{a=1}^{N} \boldsymbol{r}_a$$

This is using Cartesian coordinates. However, let's look at arbitrary generalised coordinates.

$$\begin{aligned} x_a &= x_a(q_1, q_2, \dots, q_s) \\ y_a &= y_a(q_1, q_2, \dots, q_s) \qquad\qquad S = 3N \\ z_a &= z_a(q_1, q_2, \dots, q_s) \end{aligned}$$

$$\begin{aligned} \dot{x}_a &= \sum_{k=1}^{S} \frac{\partial x_a}{\partial q_k} \dot{q}_k \equiv \frac{\partial x_a}{\partial q_k} \dot{q}_k \\ \sum_{a=1}^{N} \frac{m_a}{2} \boldsymbol{r}_a^2 &= \frac{m_a}{2}(x_a^2 + y_a^2 + z_a^2) \\ \sum_{a=1}^{N} \frac{m_a}{2} \boldsymbol{v}_a^2 &= \frac{m_a}{2}(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2) \\ &= \sum_{a=1}^{N} \frac{m_a}{2} \left(\frac{\partial x_a}{\partial q_k} \dot{q}_k \frac{\partial x_a}{\partial q_n} \dot{q}_n + \frac{\partial y_a}{\partial q_k} \dot{q}_k \frac{\partial y_a}{\partial q_n} \dot{q}_n + \frac{\partial z_a}{\partial q_k} \dot{q}_k \frac{\partial z_a}{\partial q_n} \dot{q}_n \right) \\ &= \sum_{a=1}^{N} \frac{m_a}{2} \left(\frac{\partial x_a}{\partial q_k} \frac{\partial x_a}{\partial q_n} + \frac{\partial y_a}{\partial q_k} \frac{\partial y_a}{\partial q_n} + \frac{\partial z_a}{\partial q_k} \frac{\partial z_a}{\partial q_n} \right) \dot{q}_k \dot{q}_n \\ &= \frac{1}{2} g_{kn} \dot{q}_k \dot{q}_n \qquad\qquad (g_{kn} = g_{nk}) \end{aligned}$$

i.e. kinetic energy can be written in the form

$$T = \sum_{a=1}^{N} \frac{m_a}{2} \boldsymbol{v}_a^2 = \frac{1}{2} g_{kn} \dot{q}_k \dot{q}_n$$

(g_{kn} is a symmetric metric). If all $m_a + 1$, g_{kn} is the metric of $\mathbb{R}^{S=3N}$ expressed in terms of q_i.

2.11 Metric: A way of measuring distance

e.g. $ds^2 = dx^2 + dy^2 + dz^2 = g_{ij} dx_i dx_j$ $(g_{ij} = \delta_{ij})$

Let

$$N = 1, \ \ T = \frac{m}{2} v^2 = \frac{m}{2} \left(\frac{ds}{dt} \right)^2 = \frac{m}{2} g_{ij} \dot{q}_i \dot{q}_j$$

2.11.1 Cylindrical Coordinates

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dz^2 \\ T &= \frac{m}{2}(\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) \end{aligned}$$

2.11.2 Spherical Coordinates

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + r^2 \sin^2\theta d\phi^2 \\ T &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\sin^2\theta\dot{\phi}^2) \end{aligned}$$

2.12 Motion in a field

Suppose we have a system A interacting with a system B.

$$\begin{aligned} L &= L_A + L_B + L_{int} \\ &= T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - U_A(q_A) - U_B(q_B) - U_{int}(q_A, q_B) \end{aligned}$$

Let's assume that the back reaction of system B on the action of A is negligible, therefore B executes a given motion, it moves as if A does not exist. Then, we set a Lagrangian of A in an external field created by B.

$$\begin{aligned} L_A &= T_A(q_A, \dot{q_A}) - U_A(q_A) - U_{int}(q_A, q_B(t)) \qquad q_B(t) \text{ are given coordinates of B} \\ &= T_A(q_A, \dot{q}_A) - U(q_A, q_B(t)) \end{aligned}$$

So our Lagrangian for the system A+B is (from before)

$$\begin{aligned} L &= T_A(q_A, \dot{q}_A) - U_A(q_A) - U_{int}(q_A, q_B(t)) \\ &= T_A(q_A, \dot{q}_A) - U(q_A, q_B(t)) \end{aligned}$$

2.12.1 Steps to find Lagrangian for constrained motion

1. $$L = L_{nonconstrained}$$

   S degrees of freedom; $q_1, q_2, \ldots, q_s$

2. $$C_\alpha(q_1, q_2, \ldots, q_s) = 0, \qquad \alpha = 1, \ldots, r$$

3. $y_1, y_2, \ldots, y_{s-r} \leftarrow$ independent coordinates

   $$\begin{aligned} q_1 &= q_1(y_1, y_2, \ldots, y_{s-r}) \\ &\vdots \\ q_s &= q_s(y_1, y_2, \ldots, y_{s-r}) \end{aligned}$$

4. Substitute q_i; $i = 1, \ldots, s$ into $L_{nonconstrained}$

   $$L_{constrained}(y_1, y_2, \ldots, y_{s-r}, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_{s-r}) = L_{nonconstrained}(q(y_1, y_2, \ldots, y_{s-r}), \ldots)$$

Example:

A coplanar double pendulum in a uniform gravitational field.

1. $$L_{nonconst} = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) + m_1 g y_1 + m_2 g y_2$$

2. Constraints:
$$\begin{aligned} x_1^2 + y_1^2 - l_1^2 &= 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2 &= 0 \end{aligned}$$

3. We now need to find good generalised coordinates. We use polar coordinates.
$$\begin{aligned} x_1 &= l_1 \sin\phi_1; & y_1 &= l_1 \cos\phi_1 \\ (x_2 - x_1) &= l_2 \sin\phi_2; & (y_2 - y_1) &= l_2 \cos\phi_2 \end{aligned}$$
ϕ_1 and ϕ_2 are generalised coordinates for m_1 and m_2 respectively.

4. $$\begin{aligned} &\frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) = \frac{m_1}{2}(\dot{l}_1^2 + l_1^2\dot{\phi}_1^2) = \frac{m_1}{2} l_1^2 \dot{\phi}_1^2 \\ &\frac{m_2}{2}(\dot{x}_1^2 + \dot{y}_1^2) = (l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2)^2 + (l_1 \sin\phi_1 \dot{\phi}_1 + l_2 \sin\phi_2 \dot{\phi}_2) \\ &\qquad\qquad\qquad = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) \dot{\phi}_1 \dot{\phi}_2 \end{aligned}$$

So:

$$L_{const} = \frac{m_1}{2} l_1^2 \dot{\phi}_1^2 + \frac{m_2}{2}\left[l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 (\cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \right]$$

3 Conservation Laws

3.1 Integrals of Motion

Suppose we have the initial conditions $q_i(t_0), \dot{q}_i(t_0)$.

Integrals of motion do not change in time:

$$\frac{d}{dt} I(q, \dot{q}, t) = 0$$

I depends on $q_i(t_0), \dot{q}_i(t_0)$. Any $q_i(t_0)$ or $\dot{q}_i(t_0)$ is an integral of motion.

$$\left\{ \begin{array}{ll} q_i & = q_i(q_i^0, \dot{q}_i^0, t) \\ \dot{q}_i & = \dot{q}_i(q_i^0, \dot{q}_i^0, t) \end{array} \right. \implies \begin{array}{ll} q_i^0 & = q_i^0(q_i, \dot{q}_i, t) \\ \dot{q}_i^0 & = \dot{q}_i^0(q_i, \dot{q}_i, t) \end{array}$$

In general there exist $2S$ integrals of motion.

3.1.1 Additive Integrals

Suppose we have a system composed of multiple particles. Then an additive integral of motion is one such that

$$I = \sum_{a=1}^{N} I_a$$

where we sum over the particles of the system. I is then the conserved quantities or conserved charges, e.g. electron charge. These additive integrals are important because we can reduce problems to simpler algebraic problems without solving equations of motion by using conserved quantities.

3.2 Noether's Theorem

Simply put, Noether's Theorem states that for every symmetry of our mechanical system we have a conservation law and vice versa. Let $i = 1, 2, \ldots, S$, q^i are generalised coordinates.

Greek letters α, β denote a 'generalised' index, and ε_α is a constant infinitesimal parameter of a symmetry transformation.

Examples:

$$\alpha = 1 \implies \varepsilon_1 \equiv \varepsilon$$

$$\alpha = i \implies \varepsilon_i \ ; \ i = 1, \ldots, s \text{ parameters}$$

i.e. we can parameterise our transformation with s parameters.

$$\alpha = [ij]$$

Symmetry and anti-symmetry

$\alpha = [ij]$ means that it is anti-symmetric with respect to $i \leftrightarrow j$ i.e.

$$\varepsilon[ij] = -\varepsilon[ji]$$

If we display ε as a matrix:

$$\begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{14} \\ -\varepsilon_{12} & 0 & \varepsilon_{23} & \varepsilon_{24} \\ -\varepsilon_{13} & -\varepsilon_{23} & 0 & \varepsilon_{34} \\ -\varepsilon_{14} & -\varepsilon_{24} & -\varepsilon_{34} & 0 \end{pmatrix}$$

it is clear that the number of paramaters is given by $s^2 - (\frac{s^2}{2} + \frac{s}{2}) = \frac{s(s-1)}{2}$ In the symmetric case $\varepsilon_{ij} = \varepsilon_{ji}$ we write $\varepsilon\{ij\} = \varepsilon\{ji\}$ and we have $\frac{s(s+1)}{2}$ parameters. Note that $s^2 = \#_{[ij]} + \#_{\{ij\}}$

Note we are working with a simplified version of Noether's Theorem. Suppose we have an infinitesimal transformation of generalised coordinates such that

$$q_i \to \tilde{q}_i = q_i + \sum_{\{\alpha\}} q^{i\alpha}(q,\dot{q},t)\mathcal{E}_\alpha \ + \text{ terms vanishing on-shell}^4$$

where $q^{i\alpha}(q,\dot{q},t)$ are arbitrary functions of $q, \dot{q}, t$ For example:

$$i = 1 \quad q^1 \to \tilde{q}^1 = q^1 + \textstyle\sum_{\{\alpha\}} q^{1\alpha}(q,\dot{q},t)$$

and under this transformation

$$\begin{aligned} \delta L &= \sum_{\{\alpha\}} \frac{d}{dt} L(q,\dot{q},t)\mathcal{E}_\alpha; \qquad & \delta L = L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q,\dot{q},t) \\ \delta q^i &= \sum_{\{\alpha\}} q^{i\alpha}(q,\dot{q},t)\mathcal{E}_\alpha & + \text{ terms vanishing on-shell} \end{aligned}$$

Then it is a symmetry transformation and

$$J^\alpha = \sum_{i=1}^{S} \frac{\partial L}{\partial \dot{q}^i} q^{i\alpha} - L^\alpha$$

are conserved, i.e.

$$\frac{dJ^\alpha}{dt} = 0 \text{ on-shell}$$

Example: Generalised momentum

Generalised momentum: $p_i \equiv \frac{\partial L}{\partial \dot{q}^i}$

$$\begin{aligned} L &= \sum_a \frac{m_a}{2} \boldsymbol{r}_a^2 - U(\boldsymbol{r}_a - \boldsymbol{r}_b) \\ \boldsymbol{r}_a - \boldsymbol{r}'_a &= \boldsymbol{r} + \boldsymbol{\mathcal{E}} \\ \boldsymbol{r}_a &= (x_a, y_a, z_a) \end{aligned}$$

We now check if the Lagrangian transforms as described by Noether's Theorem under the transformation

$$\begin{aligned} \bar{\mathcal{E}} &= (\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z) \\ \tilde{x}_a &= x + \mathcal{E}_x \\ \tilde{y}_a &= y + \mathcal{E}_y \\ \tilde{z}_a &= z + \mathcal{E}_z \end{aligned}$$

Under this transformation, $L \to L$. All we need now is to specify $q^{i\alpha}$.

$$q^i \to \tilde{q}^i = q^i + \sum_{\{\alpha\}} q^{i\alpha}(q,\dot{q},t)\mathcal{E}_\alpha \ + \ \text{ terms on-shell}$$

---

[4] on-shell = on solutions to equations of motion, or if $q(t)$ is a solution of equation of motion

Under this,

$$\begin{aligned}
\delta L &= \sum_{\{\alpha\}} \frac{d}{dt} L^\alpha(q, \dot{q}, t)\varepsilon_\alpha \\
J^\alpha &= \sum_{i=1}^{S} \frac{\partial L}{\partial \dot{q}^i} q^{i\alpha} - L^\alpha = \sum_{i=1}^{S} p_i q^{i\alpha} - L^\alpha \\
\dot{J}_\alpha &= 0
\end{aligned}$$

Let $\delta L = 0$

$$J^\alpha = \sum_{i=1}^{S} p_i q^{i\alpha}$$

$$\sum_{\{\alpha\}} \left(J^\alpha \varepsilon_\alpha = \underbrace{\sum_{i=1}^{S} p_i q^{i\alpha} \varepsilon_\alpha}_{\delta q^i = \tilde{q}^i - q^i} \right)$$

$$\implies \sum_{\{\alpha\}} J^\alpha \varepsilon_\alpha = \sum_{i=1}^{S} p_i \delta q^i$$

Combine ε_α into a vector $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}; \qquad J^\alpha \to \boldsymbol{J}$$

$$\boxed{\boldsymbol{J} \cdot \boldsymbol{\varepsilon} = \sum_{i=1}^{S} p_i \delta q^i}$$

Linear momentum:

$$\begin{aligned}
L &= \sum_a \frac{m_a}{2} \dot{\boldsymbol{r}}_a^2 - U(\boldsymbol{r}_a - \boldsymbol{r}_b) \\
\boldsymbol{r}_a &\to \tilde{\boldsymbol{r}}_a = \boldsymbol{r} + \boldsymbol{\varepsilon} \\
\implies \delta \boldsymbol{r}_a &= \varepsilon; \qquad \delta L = 0
\end{aligned}$$

So:

$$\begin{aligned}
\boldsymbol{J} \cdot \boldsymbol{\varepsilon} &= \sum_{i=1}^{S} p_i \delta q^i = \sum_{a=1}^{N} p_a \cdot \delta \boldsymbol{r}_a \\
&= \left(\sum_{a=1}^{N} \boldsymbol{p}_a \right) \cdot \boldsymbol{\varepsilon}
\end{aligned}$$

So:

$$\boxed{\boldsymbol{J} = \boldsymbol{p} = \sum_{a=1}^{N} \boldsymbol{p}_a}$$

i.e. we find that when we have homogeneity of space (translational symmetry) then momentum is conserved.

3.3 Proof of Noether's Theorem

We have an action:

$$S = \int_{t_1}^{t_2} L \, dt$$

Consider the variation:

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \delta L \, dt \\ &= \int_{t_1}^{t_2} \frac{d}{dt} J^\alpha \mathcal{E}_\alpha \, dt \qquad (1) \end{aligned}$$

Now:

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt \\ &= \int_{t_1}^{t_2} \underbrace{\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right)}_{=0} \delta q^i \, dt + \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) dt \end{aligned}$$

So:

$$So : \delta S = \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) dt \qquad (2)$$

Thus, combining (1) and (2):

$$\begin{aligned} \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i - J^\alpha \mathcal{E}_\alpha \right) dt = 0 \qquad \forall t_1, t_2, \mathcal{E}_\alpha \\ J^\alpha(t_2) \mathcal{E}_\alpha - J^\alpha(t_1) \mathcal{E}_\alpha = 0 \\ J^\alpha(t_2) = J^\alpha(t_1) = J^\alpha \end{aligned}$$

□

Example: Cyclic Coordinates

Cyclic coordinates (isometry of the configuration space) $\equiv$ the space of q^i.

Isometry: $q^1 \to q^1 + \varepsilon$ We want the Lagrangian to be invariant under this transformation, i.e. $\delta L = 0 \implies L(q_2, q_3, \ldots, q_S, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_S, t)$ (the Lagrangian has no explicit dependence on q^1)

$$\begin{aligned} \delta q^1 &= \varepsilon_1, = \delta q^2 = 0 = \delta q^3 = \ldots = \delta q^S \\ J &= \frac{\partial L}{\partial \dot{q}^i}\delta q^i = \frac{\partial L}{\partial \dot{q}^1}\varepsilon = p_1 \cdot \varepsilon \\ J &= p_1 \leftarrow \text{generalised momentum} \end{aligned}$$

Explanation: We find that if the Lagrangian is invariant under an infinitesimal translation along q^1 (i.e. we have isometry) then the Lagrangian will have no explicit dependence on q^1. From this we can see that the generalised momentum corresponding to qi is invariant, i.e. we have a conservation law. Coordinates like q^1 which do not appear explicitly in the Lagrangian are called cyclic coordinates.

3.4 Linear Momentum

$$\left.\begin{aligned} L = \frac{m_a}{2}\boldsymbol{v}^2 - U(\boldsymbol{r}_a - \boldsymbol{r}_b) \\ \delta \boldsymbol{r}_a, = \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} = (\varepsilon, 0, 0) \end{aligned}\right\} \qquad \text{What coordinate system can we choose for this?}$$

$$\boldsymbol{R} = \frac{\sum\limits_a m_a \boldsymbol{r}_a}{\sum\limits_b m_b} \qquad \text{centre of mass}$$

$$\delta \boldsymbol{R} = \boldsymbol{\varepsilon} \qquad \text{R is the cyclic coordinate in this case}$$

$$\delta R_x = \varepsilon_1, \qquad \delta R_y = \ldots = \delta R_z = 0$$

Example: Conservation of energy (homogeneity of time)

$$\begin{aligned}
t &\to t + \varepsilon \\
q^i t &\to \tilde{q}^i(t) = q^i(t+\varepsilon) = q^i(t) + \dot{q}^i(t)\varepsilon \\
\delta q_i &= \dot{q}_i \cdot \varepsilon \\
L(q,\dot{q},t) &\to \tilde{L}(q+\delta q, \dot{q}+\delta\dot{q}, t) \\
&= L(q,\dot{q},t) + \frac{\partial L}{\partial q^i}\delta q + \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q} \\
&\implies \delta L(q,\dot{q},t) = \frac{\partial L}{\partial q^i}\dot{q}^i\varepsilon + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i\varepsilon \\
\delta L &= \frac{d}{dt}(L\varepsilon) - \frac{\partial L}{\partial t}\varepsilon
\end{aligned}$$

We require that $\frac{\partial L}{\partial t}\varepsilon$ vanish (so that the two Lagrangians differ only by a total time derivative), which means the Lagrangian has no explicit dependence on time, i.e. $L(q,\dot{q})$.

$$\begin{aligned}
\frac{d}{dt}L\left(q(t),\dot{q}(t),t\right) &= \frac{\partial L}{\partial q}\cdot\dot{q} + \frac{\partial L}{\partial \dot{q}}\cdot\ddot{q} + \frac{\partial L}{\partial t} \\
\delta L &= \frac{d}{dt}(L\varepsilon) \implies L' = L \\
J' &= \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = E \\
E &= p_i\dot{q}^i - L \\
L &= \sum_{i=1}^{S}\frac{m_i}{2}\dot{q}_i^2 - U(q) \\
E &= \sum_{i=1}^{S} m_i\dot{q}_i\cdot\dot{q}_i - \left(\sum_{i=1}^{S}\frac{m_i}{2}\dot{q}_i^2 - U\right) \\
&= \sum_{i=1}^{S}\frac{m_i}{2}\dot{q}_i^2 + U(q) \\
\boldsymbol{p} &= \sum_{a=1}^{N}\boldsymbol{p_a} \qquad \leftarrow \text{additive}
\end{aligned}$$

but

$$\boldsymbol{E} = \sum_{a=1}^{N}\frac{m_a}{2}\boldsymbol{v}_a^2 + U(\boldsymbol{r}_a) \neq \sum_{a=1}^{N}E_a \qquad \leftarrow \text{not additive}$$

however, if $|\boldsymbol{r}_a - \boldsymbol{r}_b| \to \infty$, $U(\boldsymbol{r}_a) \to 0$ then

$$E = \sum_{a=1}^{N} T_a^{\infty}$$

3.5 Angular momentum / Isotropy of space

Assume $\delta L = 0$ under rotations. $\delta\phi$ is the vector of the infinitesimal rotation. $\delta\phi = |\delta\boldsymbol{\phi}|$ is the angle of rotation.

$$
\begin{aligned}
\delta\phi &\to \boldsymbol{\varepsilon} \\
\delta\boldsymbol{r}_a &= \delta\boldsymbol{\phi} \times \boldsymbol{r}_a \\
\boldsymbol{J} \cdot \boldsymbol{\varepsilon} &= \sum_{a=1}^{N} \frac{\partial L}{\partial \boldsymbol{v}} \cdot \delta\boldsymbol{r}_a = \boldsymbol{J} \cdot \delta\boldsymbol{\phi} \\
&= \sum_{a=1}^{N} \boldsymbol{p}_a \cdot (\delta\boldsymbol{\phi} \times \boldsymbol{r}_a) \\
&= \sum_{a=1}^{N} \delta\boldsymbol{\phi} \cdot (\boldsymbol{r}_a \times \boldsymbol{p}_a) \qquad \text{(since $\times$ exhibits cyclical symmetry)} \\
&\implies \boldsymbol{J} \equiv \boldsymbol{M} = \sum_{a=1}^{N} \boldsymbol{r}_a \times \boldsymbol{p}_a
\end{aligned}
$$

i.e. we find that the quantity $M = \sum_{a=1}^{N} \boldsymbol{r}_a \times \boldsymbol{p}_a$, which we call the angular momentum, is conserved if the Lagrangian is such that it is unchanged under an infinitesimal rotation.

3.5.1 The Levi-Civita symbol

The Levi-Civita symbol $\varepsilon_{\alpha\beta\gamma}$ is defined for $\alpha, \beta, \gamma \in \{1, 2, 3\}$ such that $\varepsilon_{123} = 1$ and such that it is skew-symmetric under exchange of any pair of indices, i.e. $\varepsilon_{\alpha\beta\gamma} = -\varepsilon_{\beta\alpha\gamma} = -\varepsilon_{\alpha\gamma\beta} = \ldots$. This allows us to work out the value of the symbol for any α, β, γ. The number of non-vanishing components is $3! = 6$, i.e. $\varepsilon_{111} = \varepsilon_{212} = 0$ etc. It can be seen that the symbol is positive for an even permutation and negative for an odd permutation.

3.5.2 Applicability to cross products

The following statements are equivalent:

$$
\boxed{\begin{aligned} \boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{b} \\ c_\alpha &= \varepsilon_{\alpha\beta\gamma} a_\beta b_\gamma \end{aligned}} \qquad \leftarrow \text{sum over repeated indiced } \beta, \gamma
$$

This can be proven easily by computing directly the elements of $\boldsymbol{c}$, e.g.

$$
c_2 = \varepsilon_{2\beta\gamma} a_\beta b_\gamma = \varepsilon_{213} a_1 b_3 + \varepsilon_{231} a_3 b_1 = -a_1 b_3 + a_3 b_1
$$

We can easily derive some useful relations for the cross product using the Levi Civita symbol, for instance:

$$\begin{aligned}
\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= c_\alpha(\boldsymbol{a}\times\boldsymbol{b})_\alpha \\
&= \varepsilon_{\alpha\beta\gamma}c_\alpha a_\beta b_\gamma \\
&= \varepsilon_{\beta\gamma\alpha}a_\beta b_\gamma c_\alpha \\
&= a_\beta\varepsilon_{\beta\gamma\alpha}b_\gamma c_\alpha \\
&= a_\beta(\boldsymbol{b}\times\boldsymbol{c})_\beta \\
&= \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \\
&\therefore \\
\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})
\end{aligned}$$

3.5.3 Properties of Angular Momentum

Here we show that the angular momentum for a system of particles is additive. First:

$$\begin{aligned}
\delta\boldsymbol{r} &= \delta\boldsymbol{\phi}\times\boldsymbol{r} \\
\delta x_{\alpha a} &= \varepsilon_{\alpha\beta\gamma}\delta\phi_\beta x_{\gamma a}
\end{aligned}$$

Now, we compute the total angular momentum about our axis of rotation:

$$\begin{aligned}
\delta\phi_\beta\cdot M_\beta &= \sum_{a,\alpha}\frac{\partial L}{\partial\dot{x}_{\alpha a}}\cdot\delta x_{\alpha a} \\
&= \sum_{a,\alpha}p_{\alpha a}\cdot\varepsilon_{\alpha\beta\gamma}x_{\gamma\alpha}\delta\phi_\beta
\end{aligned}$$

So:

$$\begin{aligned}
M_\beta &= \sum_\alpha p_{\alpha a}\varepsilon_{\alpha\beta\gamma}x_{\gamma\alpha} \\
&= \sum_\alpha \varepsilon_{\alpha\beta\gamma}x_{\gamma\alpha}p_{\gamma\alpha} \\
&= \sum_a(\boldsymbol{r_a}\times\boldsymbol{p_a})_\beta \\
\boldsymbol{M} &= \sum_a\underbrace{\boldsymbol{r}_a\times\boldsymbol{p}_a}_{M_a} \qquad \leftarrow \text{additive}
\end{aligned}$$

So we see that the total angular momentum of a system is the sum of the angular momenta of the parts. We can also show that $\boldsymbol{M}$ depends on the choice of origin:

Let:

$$\boldsymbol{r}_a = \boldsymbol{b} + \boldsymbol{r}'_a$$

So:

$$\begin{aligned}\boldsymbol{M} &= \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a = \sum_a \boldsymbol{r}'_a \times \boldsymbol{p}_a + \boldsymbol{b} \times \sum_a \boldsymbol{p}_a \\ &= \boldsymbol{M}' + \boldsymbol{b} \times \boldsymbol{p}\end{aligned}$$

Finally we note that if an external field is symmetric about some axis, the component of M along the axis of symmetry is conserved.

4 Lie Groups

This section is an aside and isn't directly related to the core material.

4.1 Group definition reminder

A group is a set G, together with an operation '$\cdot$' called multiplication, or the group law of G, which assigns to any two elements of the set a third set such that we have:

1. *Closure:* $\forall a, b, \in G,\ a \cdot b \in G$
2. *Associativity:* $\forall a, b, c \in G,\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. An *identity element,* id: $\exists\ id \in G : \forall a \in G,\ id \cdot a = a \cdot id = a$
4. *Inverses:* $\forall a \in G\ \exists\ a^{-1} \in G : a \cdot a^{-1} = a^{-1} \cdot a = id$

Remark: In general $a \cdot b \neq b \cdot a$. If $a \cdot b = b \cdot a$ the group is called *abelian.*

4.1.1 Example: Integers

The set of integers: $0, \pm 1, \pm 2, \ldots$ denoted by $\mathbb{Z}$ with $\cdot \equiv +$. Then

$$\begin{aligned} a \cdot b &\equiv a + b \\ id &\equiv 0 \\ a^{-1} &\equiv -a \\ a + b &= b + a; \qquad \mathbb{Z} \text{ is abelian}\end{aligned}$$

4.1.2 Example: The cyclic group

A set of n elements: $id, \omega, \omega^2, \ldots, \omega^{n-1}, \omega^n = id$ denoted by $\mathbb{Z}_n$ called the cyclic group.

e.g.

$$id = 1; \qquad \omega = e^{2\pi i/n}; \qquad \omega^k = e^{k(2\pi i/n)}; \qquad \omega^n = e^{2\pi i} = 1$$

4.1.3 Example: The group of permutations

The group of permutations of 1,2,3 is called S_3.

$$p_{ijk} = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix};$$

$$p_{ijk}{}^{-1} = \begin{pmatrix} i & j & k \\ 1 & 2 & 3 \end{pmatrix};$$

Permutations of $1, 2, \ldots, n$ is S_n.
The number of elements is $n!$.
S_n for $n \geq 3$ is non-abelian. Note that $S_2 = \mathbb{Z}_2$.

4.2 Lie groups

Lie Groups are groups which are manifolds. A manifold is a higher dimension generalisation of curves (1-d manifold) and surfaces (2-d manifold). It is a curved space. Lie groups are useful for analysing continuous symmetries, like the ones we have seen in the previous sections.

4.2.1 Example: The real line

The real line $\mathbb{R}$ is a Lie Group.

$$\cdot \equiv +; \qquad x + y \in \mathbb{R}; \qquad id \equiv 0$$

4.2.2 Example: The circle

The circle S^1 is a Lie group we call U(1). It is abelian.

$$e^{i\varphi_1} \cdot e^{i\varphi_2} = e^{i(\varphi_1 + \varphi_2)}$$

$$U(1) \approx SO(2) \qquad \text{(group of rotations in 2-d space)}$$

$$A \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \qquad \begin{aligned} A \cdot A^T &= I \\ \det A &= 1 \end{aligned}$$

4.2.3 Example: The special orthogonal group SO(D)

The special orthogonal group, denoted $SO(D)$ is a matrix group.

$$SO(D): \left.\begin{array}{r} A\cdot A^T = I \implies A^{-1} = A^T \\ \det A = 1 \end{array}\right\} \text{the two requirements of SO(D)}$$

The 'S' of SOD requires that $\det A = 1$.

$$A_1A_2\cdot(A_1A_2)^T = A_1A_2A_2^TA_1^T = I$$

The group satisfies the quadratic form $x_1^2 + x_2^2 + \ldots + x_D^2 = x_D'^2$ where $x_D'^2$ is a rotated vector, i.e. $A : A\cdot x = x'$

4.2.4 Example: The special unitary group SU(N)

The special unitary group, denoted $SU(N)$ is a matrix group. It consists of the set of complex valued matrices A such that $A\cdot A^+ = I$, $\det A = 1$, where A^+ is the conjugate transpose of A.

For $N = 2, SU(2)$, it must meet the following requirements:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = I; \qquad ad - bc = 1 \qquad (\det A = 1)$$

Solution:

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2), \qquad a\bar{a} + b\bar{b} = 1$$

$$\left.\begin{array}{r} a = x + iy \\ b = z + iw \end{array}\right\} \qquad x^2 + y^2 + z^2 + w^2 = 1 \qquad \text{(3-dimensional sphere)}$$

So in other words, the points of a 3-sphere (not to be confused with the 2-sphere, which is the familiar 2-dimensional surface embedded in 3-dimensional space), form a group under the $SU(2)$ group law described above.

5 Roller Coaster Problem

In this section we look at points whose motion are constrained to 3-dimensional curves. We can describe such curves with parametric equations:

$$x = x(q), \qquad y = y(q), \qquad z = z(q)$$

where q is a generalised coordinates. e.g. for a helix,

$$x = R\cos q, \qquad y = R\sin q, \qquad z = vq$$

So then, if we have our Lagrangian:

$$\begin{aligned} L &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{m}{2}\underbrace{(x_2' + y_2' + z_2')}_{\text{we denote as } a(q)}\dot{q}^2 - mgz(q) \end{aligned}$$

So:

$$L = \frac{m}{2}a(q)\dot{q}^2 - U(q)$$

which is the most general one dimensional Lagrangian.

We introduce the *arc-length parameter*:

$$\dot{S} = \sqrt{a(q)}\dot{q} \qquad \text{or} \qquad ds = \sqrt{a(q)}dq$$

Integrating:

$$S = \int\limits_{q_0}^{q} \sqrt{a(\tilde{q})}d\tilde{q} = S(q)$$

Solve for $q : q = q(S)$ so:

$$L = \frac{m}{2}\dot{S}^2 - U(S)$$

where:

$$U(S) = mgz(q(S))$$

5.1 One-dimensional case

So consider the example (picture) where we have some particle constrained to move along some curve in the xz-plane:

$$\begin{aligned} z &= f(x), \\ L &= \frac{m}{2}(\dot{x}^2 + \dot{z}^2) - mgf(x) \\ &= \frac{m}{2}(1 + f'^2)\dot{x}^2 - mgf(x) \end{aligned}$$

then:

$$S = \int\limits_{x_0}^{x} \sqrt{1 + f'^2(\tilde{x})}d\tilde{x} = S(x)$$

Note that S is an increasing function of x, i.e. $\dot{S} = \sqrt{1 + f'^2} > 0$. We have:

$$L = \frac{m}{2}\dot{S}^2 - mgf(x(S))$$

Note that $F = \frac{\partial U}{\partial S} = mgf'(x)x'(S)$. For clarity we revert back to the variable x here, i.e. $S \to x$ so that:

$$L = \frac{m}{2}\dot{x}^2 - U(x)$$

We wish to find x as a function of time. We do this using conservation of energy, since time isn't part of our function:

$$E = \frac{m}{2}\dot{x}^2 + U(x) \qquad \leftarrow \text{conserved}^5$$

so:

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - U(x))}$$

let $x > 0$ so:

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}$$

so:

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}} + \text{const. (determined by initial conditions)}$$

We know t is a function of x, therefore x is a function of t too, i.e. $t = t(x) \implies x = x(t)$

TODO: Insert picture We can see that if the starting point of the particle is between x_A and x_B the particle will oscillate between these two points, thus they are turning points. Between x_A, x_B the motion is finite. We call this region a potential well.

$$t = \sqrt{\frac{m}{2}} \int\limits_{x_A(E)}^{x} \frac{dx}{\sqrt{E - U(x)}}; \qquad\qquad x > 0$$

thus:

$$T = \sqrt{\frac{m}{2}} \int\limits_{x_A(E)}^{x_B(E)} \frac{dx}{\sqrt{E - U(x)}}$$

Thus we have derived the equation for the period of oscillation of a particle oscillating between two points x_A and x_B in a potential well. This is a very useful and general result. We can apply this to a common example:

---

[5] Remember from the discussion on the conservation of energy that if the Lagrangian is written $L = T - U$ then $E = T + U$.

5.1.1 Example: simple pendulum

The Lagrangian for a simple pendulum is given by the formula $L = \frac{ml^2}{2}\dot{\phi}^2 - mgl\cos\phi$. Let ϕ_0 be the maximum angle. Then:

$$E = mgl\cos\phi_0 = \frac{ml^2}{2}\dot{\phi}^2 + mgl\cos\phi$$

so:

$$\begin{aligned} T &= 2\sqrt{\frac{ml^2}{2}} \int\limits_{-\phi_0}^{\phi_0} \frac{d\phi}{\sqrt{mgl\cos\phi_0 - mgl\cos\phi}} \\ &= 4\sqrt{\frac{l}{2g}} \int\limits_{0}^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi_0 - \cos\phi}} \\ &= 2\sqrt{\frac{l}{g}} \int\limits_{0}^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}} \end{aligned}$$

Now, let $\sin\xi = \frac{\sin(\phi/2)}{\sin(\phi_0/2)}$ so that[6]:

$$T = 4\sqrt{\frac{l}{g}}K(sin\frac{\phi_0}{2}) \qquad \text{where} \qquad \underbrace{K(z) = \int\limits_{0}^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - z^2\sin^2\xi}}}_{\text{complete elliptic integral of the 1st kind}}$$

then[7], for $\phi_0 \ll 1$ we find that $T \approxeq 2\pi\sqrt{\frac{l}{g}}$, the usual formula for the period of a simple pendulum.

---

[6] $\xi = \arcsin\left(\frac{\sin(\phi/2)}{\sin(\phi_0/2)}\right)$ so:

$$\frac{d\xi}{d\phi} = \frac{\cos(\phi/2)}{2\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}} = \frac{\sqrt{1 - \sin^2(\phi/2)}}{2\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}}$$

thus:

$$d\phi = 2\left(\frac{\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}}{\sqrt{1 - \sin^2(\phi/2)}}\right)d\xi$$

and the rest should be clear.

[7] This follows easily from noticing that $sin(\theta/2) \approxeq (\theta/2)$ for small θ and doing a Taylor expansion of K

5.2 Determining the potential energy from the period of oscillation

$$T(E) = \sqrt{2m} \int\limits_{x_A}^{x_B} \frac{dx}{\sqrt{E - U(x)}}$$

This is in a sense the inverse problem of what we did before. We can view x_1 and x_2 as functions of U since we divide it into sections where the function doesn't change direction:

$$= \sqrt{2m} \int\limits_{x_1(E)}^{0} \frac{dx}{\sqrt{E - U(x)}} + \sqrt{2m} \int\limits_{0}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}$$

Now, $z = U(x)$, $\qquad x = x_1(z)$, $\qquad dx = \frac{dx_1}{dz} dz$ $\qquad$ (and similarly for x_2) so:

$$\begin{aligned} &= \sqrt{2m} \int\limits_{E}^{0} \frac{\frac{dx_1}{dz} dz}{\sqrt{E - z}} + \sqrt{2m} \int\limits_{0}^{E} \frac{\frac{dx_2}{dz} dz}{\sqrt{E - z}} \\ &= \sqrt{2m} \int\limits_{0}^{E} \frac{\left(\frac{dx_2}{dz} - \frac{dx_1}{dz} \right)}{\sqrt{E - z}} dz \end{aligned}$$

dividing both sides by $\sqrt{W - E}$ (where W is a parameter) and integrating:

$$\begin{aligned} \int\limits_{0}^{W} \frac{T(E)}{\sqrt{W - E}} dE &= \sqrt{2m} \int\limits_{0}^{W} \int\limits_{0}^{E} \left(\frac{dx_2}{dz} - \frac{dx_1}{dz} \right) \frac{dz}{\sqrt{E - z}} \frac{1}{\sqrt{W - E}} dE \\ &= \sqrt{2m} \int\limits_{0}^{W} \int\limits_{Z}^{W} \left(\frac{dx_1}{dz} - \frac{dx_2}{dz} \right) \frac{dE}{\sqrt{E - z}} \frac{dz}{\sqrt{W - E}} \qquad (1) \end{aligned}$$

Now, note that[8] $(E - z)(W - E) = -\left(E - \frac{W+z}{2}\right)^2 + \left(\frac{W-z}{2}\right)$ and let $\alpha = E - \frac{W+z}{2}$:

$$\int\limits_{Z}^{W} \frac{dE}{\sqrt{(E - z)(W - E)}} = \int\limits_{\frac{z-W}{2}}^{\frac{W-z}{2}} \frac{d\alpha}{\sqrt{\left(\frac{w-z}{2}\right) - \alpha^2}}$$

then, let $\alpha = \frac{W-z}{2} \beta$:

$$= \int\limits_{-1}^{1} \frac{\left(\frac{W-z}{2}\right) d\beta}{\sqrt{\left(\frac{W-z}{2}\right)^2 (1 - \beta^2)}} = 2 \int\limits_{0}^{1} \frac{d\beta}{\sqrt{1 - \beta^2}}$$

and finally, let $\beta = \sin\phi$ so $d\beta = \cos\phi d\phi$

$$= 2\int\limits_0^{\pi/2} \frac{\cos\phi d\phi}{\sqrt{1-\sin^2\phi}} = 2\int\limits_0^{\pi/2} d\phi = \pi$$

So the RHS of our integral (1) reduces to

$$\sqrt{2m}\int\limits_0^W \left(\frac{dx_2}{dz} - \frac{dx_1}{dz}\right)\cdot \pi dz = \pi\sqrt{2m}\left(x_2(W) - x_1(W)\right)$$

Changing our variable back from W to U we arrive at our result:

$$\boxed{x_2(U) - x_1(U) = \frac{1}{\pi\sqrt{2m}}\int\limits_0^U \frac{T(E)}{\sqrt{U-E}}dE}$$

Thus we can use the known function $T(E)$ to determine the difference $x_2(U) - x_1(U)$. $x_2(U)$ and $x_1(U)$ themselves are indeterminate, however if we impose the condition that the curve $U = U(x)$ is symmetrical about the $U-$axis, then we have $x2(U) = -x_1(U) \equiv x(U)$, and we have the expression

$$\boxed{x(U) = \frac{1}{2\pi\sqrt{2m}}\int\limits_0^U \frac{T(E)}{\sqrt{U-E}}dE}$$

6 Motion in a central field

6.1 Case Study: A scattering process

Consider the system of two particles described in the picture. The Lagrangian is $L = \frac{m_1}{2}v^2 + \frac{m_2}{2}v^2 - U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$ We have:

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2; \qquad \boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$$

so:

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1+m_2}\boldsymbol{r}; \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1+m_2}\boldsymbol{r}$$

Thus:

$$\begin{aligned}\frac{m_1}{2}\dot{\boldsymbol{r}}_1^2 + \frac{m_2}{2}\dot{\boldsymbol{r}}_2^2 &= \frac{m_1}{2}\left(\dot{\boldsymbol{R}} + \frac{m_2}{m_1 + m_2\dot{\boldsymbol{r}}}\right)^2 + \frac{m_2}{2}\left(\dot{\boldsymbol{R}} + \frac{m_1}{m_1 + m_2\dot{\boldsymbol{r}}}\right)^2 \\ &= \frac{m_1+m_2}{2}\dot{\boldsymbol{R}}^2 + \frac{m_1m_2^2 + m_2m_1^2}{2(m_1+m_2)^2}\dot{\boldsymbol{r}}^2\end{aligned}$$

---

[8] This is shown by completing the square:

$$\begin{aligned}(E-z)(W-E) &= \left[-E^2 + (W+z)E\right] \quad - zW \\ &= \left[-\left(E - \frac{W+z}{2}\right)^2 + \left(\frac{W+z}{2}\right)^2\right] \quad - zW \quad = -\left(E - \frac{W+z}{2}\right)^2 + \left(\frac{W-z}{2}\right)\end{aligned}$$

so

$$L = \frac{m_1 + m_2}{2}\dot{\boldsymbol{R}}^2 + \underbrace{\frac{m_1 m_2}{2(m_1 + m_2)}}_{\text{reduced mass conserved}}\dot{\boldsymbol{r}}^2 \quad -U(r)$$

So six degrees of freedom is reduced to three degrees of freedom. $U(r)$ is a central field. We use the conservation of linear momentum to reduce this system.

6.2 Motion in a central field

We start with the Lagrangian:

$$L = \frac{m}{2}\dot{\boldsymbol{r}}^2 - U(r)$$

Energy and momentum in this system are conserved (spherical symmetry). Since[9] $\boldsymbol{M}$ is always conserved, the motion must always take place in the plane defined by $\boldsymbol{M}$. We introduce polar coordinates:

$$x = r\cos\theta; \qquad y = r\cos\theta$$
$$L = (\frac{m}{2}\dot{\boldsymbol{r}}^2 + r^2\dot{\phi}^2) - U(r)$$

ϕ is cyclic $\implies$ $p\phi = mr^2\dot{\phi}$ is conserved. $p\phi = M$

$$\begin{aligned}
M_z &= (\boldsymbol{r} \times \boldsymbol{p})_z \\
&= xp_y - yp_x \\
&= mr\cos\phi(\dot{r}\sin\phi + r\cos\phi\dot{\phi}) - mr\sin\phi(\dot{r}\cos\phi - r\sin\phi\dot{\phi}) \\
&= mr^2\dot{p} \\
&= p\phi
\end{aligned}$$

now,

$$mr^2\dot{\phi} = M \qquad \text{so:} \qquad \dot{\phi} = \frac{M}{mr^2} > 0$$
$$mr^2 d\phi = M dt$$

What is $mr^2 d\phi$?

$$\begin{aligned}
r^2 d\phi &= 2dA \\
dA &= \frac{M}{2m}dt
\end{aligned}$$

so:

$$\dot{A} = \frac{M}{2m} = const$$

---

[9] (Remember that $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}$ is angular momentum)

Which gives us Kepler's Second Law.

$$\begin{aligned}
E &= \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\phi}^2 + U(r) \\
&= \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\left(\frac{M}{mr^2}\right) + U(r) \\
&= \frac{m}{2}\dot{r}^2 - U_{\text{eff}}(r) \\
U_{\text{eff}} &= U(r) + \frac{M^2}{2mr^2} \qquad \leftarrow \text{centrifugal energy}
\end{aligned}$$

so:

$$\begin{aligned}
\dot{r} &= \sqrt{\frac{2}{m}\left(E - U_{\text{eff}}(r)\right)} \\
t &= \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{\text{eff}}(r)\right)}} + const. \qquad \leftarrow \text{distance } r \text{ from centre an implicit function of time}
\end{aligned}$$

now,

$$d\phi = M\frac{dt}{mr^2} = \frac{M}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{\text{eff}}(r)\right)}}$$

so:

$$\phi = \int \frac{M dr}{r^2\sqrt{2m\left(E - U_{\text{eff}}(r)\right)}} \qquad \leftarrow \text{equation of the path}$$

TODO: ADDPICTURES! Remember that U_{eff} depends on ϕ!. Doesn't oscillate $\implies$ 1-D motion $\implies$ Moves between 2 circles: TODO: PICTURE. Not a closed path. 2 closed path cases: $U(r) \approx \frac{1}{r}; U(r) \approx r^2 \to$ space oscillator.

Looking at $U_{\text{eff}} < r$

$$\Delta\phi = 2\int_{r_{\min}}^{r_{\max}} \frac{M dr}{r^2\sqrt{2m\left(E - U_{\text{eff}}(r)\right)}}$$

When is the trajectory a closed path? $\Delta\phi = 2\pi\frac{k}{n}$ where k and n are mutually simple (co-prime). Then after n revolutions back to the same point.

$U \approx \frac{1}{r}, U \approx r^2 \to x^2 + y^2 \to$ 2 decoupled harmonic oscillators of frequency ω. TODO: More pictures here.

6.2.1 Example: Motion of a spherical pendulum

A spherical pendulum is a particle constrained to move on a sphere. Our Lagrangian is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \qquad \text{and} \qquad x^2 + y^2 + z^2 = R^2$$

Switching to polar coordinates, we have:

$$\left.\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned}\right\} \qquad L = \frac{m}{2}R^2(sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) - mgR\cos\theta$$

ϕ is a cyclic coordinate,

$$\begin{aligned} p\phi &= mR^2\sin^2\theta\dot{\phi} = \text{const} = M_z \\ \implies \dot{\phi} &= \frac{M_z}{mR^2\sin^2\theta} \end{aligned}$$

therefore:

$$\begin{aligned} E &= \frac{mR^2}{2}(\sin^2\dot{\phi}^2 + \dot{\theta}^2) + mgR\cos\theta \\ &= \frac{mR^2}{2}\dot{\theta}^2 + \underbrace{mgR\cos\theta + \frac{M_z^2}{2mR^2\sin^2\theta}}_{U_{\text{eff}}} \\ \implies t &= \int\frac{d\theta}{\sqrt{\frac{2}{mr^2}\left(E - U_{\text{eff}}(\theta)\right)}} \end{aligned}$$

Between θ_1 and θ_2 the particle will oscillate, but will also rotate around the z-axis. Thus it is not in general a closed trajectory.

Thus we saw an example of a 3-dimensional problem that can be reduected to a 1-dimensional problem. TODO: picture

7 Kepler's Problem

Kepler's problem is that where the potential is given by $U \approx \frac{1}{r}$, examples of such are given by Newtonian gravitational attraction, or Coulomb electrostatic interaction (attractive and repulsive). Attractive field:

$$\begin{aligned} U &= \frac{-\alpha}{r}; & \alpha > 0 \\ U_{\text{eff}} &= \frac{-\alpha}{r} + \frac{M^2}{P2mr^2}; & r > 0 \end{aligned}$$

From the equation we can say that there is obviously only one minimum:

$$r_{min} = \frac{M^2}{m\alpha}, \qquad U_{\text{eff}min} = -\frac{m\alpha^2}{2M^2}$$

7.1 Case: E less than 0

If $E < 0$ then the motion is finite.

$$\phi(r) = \int\limits_{r_{min}}^{r} \frac{M\frac{d\tilde{r}}{\tilde{r}^2}}{\sqrt{2m(E - U_{\text{eff}})}} = \int \frac{M\frac{d\tilde{r}}{\tilde{r}^2}}{\sqrt{2m\left(E + \frac{\alpha}{\tilde{r}} - \frac{M^2}{2m\tilde{r}^2}\right)}}$$

This integral can be solved by a substitution:

$$x = \frac{1}{\tilde{r}}; \qquad r_{min} \to x_{min} = \frac{1}{r_{min}}$$
$$dx = -\frac{d\tilde{r}}{\tilde{r}^2}$$

So we have the integral, which we already now how to solve by completing the square:

$$\phi(r) = \int\limits_{1/r_{min}}^{1/r} \frac{M dx}{\sqrt{2m\left(E + \alpha x - \frac{M^2}{2m}x^2\right)}} = \arccos\frac{\frac{M}{r} - \frac{m\alpha}{M}}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}$$

so:

$$\cos\phi = \frac{\frac{M}{r} - \frac{m\alpha}{M}}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}$$

or:

$$\frac{p}{r} = 1 + e\cos\phi \qquad \leftarrow \text{equation of conic section}$$

where:

$$p = \frac{M^2}{m\alpha}; \qquad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}} \qquad \text{(show this!)}$$

$2p$ is called the *latus rectum* while e is called the *eccentricity*. The point where $\phi = 0$ is the point nearest to the centre. It is called the *perihelion*. Note that $E < 0 \implies 0 \leq e \leq 1$. Also, note that

$$r_{min} = \frac{p}{1+e}; \qquad r_{max} = \frac{p}{1-e}$$

Note that if $e > 1$, we eventually hit the point where r tends to infinity. Note that the equation of an ellipse is given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ where a and b are the major and minor semi-axes respectively, $a \geq b$. Claim:

$$a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}$$
$$b = \frac{p}{\sqrt{1-e^2}} = \frac{M}{\sqrt{2m|E|}}$$

TODO: Picture of an ellipse

We have $r = \frac{p}{1+e\cos\phi}$; $\qquad a = \frac{1}{2}(r_{min} + r_{max})$; $\qquad b = \frac{p}{\sqrt{1-e^2}}$. Thus:

$$x = a - r_{min} + r\cos\phi = \frac{pe}{1-e^2} + r\cos\phi = ae + r\cos\phi = ae + \frac{p\cos\phi}{1+e\cos\phi}$$

$$y = r\sin\phi = \frac{p\sin\phi}{1+e\cos\phi}$$

so:

$$\begin{aligned}\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= \left(e + \frac{(1-e^2)\cos\phi}{1+e\cos\phi}\right)^2 + \left(\frac{\sqrt{1-e^2}\sin\phi}{1+e\cos\phi}\right)^2 \\ &= \frac{(e + \cancel{e^2\cos\phi} + (1-\cancel{e^2})\cos\phi)^2 + (1-e^2)\sin^2\phi}{(1+e\cos\phi)^2} \\ &= \frac{\cancel{e^2} + 2e\cos\phi + \cancel{\cos^2\phi} + 1 - \cancel{e^2} - \cancel{\cos^2\phi} + e^2\cos^2\phi}{(1+e\cos\phi)^2} \\ &= 1\end{aligned}$$

Note that $2m\dot{A} = M$, so

$$\underbrace{mA}_{\text{area of ellipse}} = \underbrace{TM}_{\text{period of revolution}}$$

$$T = \frac{2mA}{M} = \frac{2m\pi ab}{M} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}$$

7.2 Case: E greater than 0

TODO: insert picture If $E > 0$ the motion is no longer finite, the motion is a hyperbolic orbit. We will have $e > 1$. Since $r = p/(1+e\cos\phi)$ we have

$$\boxed{r_{min} = \frac{p}{1+e}} = a(e-1) \quad \boxed{r_{max} = \infty} \implies \cos\phi_{max} = \frac{-1}{e} \implies \phi_{max} \geq \frac{\pi}{2}$$

7.3 Case: E is exactly 0

If $E = 0$ then we have $e = 1$. We find that the motion will be a parabolic orbit. Since $r = p/1 + \cos\phi$ we have

$$r_{min} = \frac{p}{2} \qquad\qquad E = 0 \implies v_\infty = 0$$

Time dependence:

$$\begin{aligned}t &= \sqrt{\frac{M}{2|E|}} \cdot \int \frac{rdr}{\sqrt{-r^2 + \frac{\alpha}{|E|}r - \frac{M^2}{2m|E|}}} \\ &= \frac{ma}{\alpha}\int \frac{rdr}{a^2e^2 - (r-a)^2}\end{aligned}$$

substitution:

$$r = a(1 - e\cos\zeta); \qquad t = \sqrt{\frac{ma^3}{\alpha}}(\zeta - e\sin\zeta)$$

7.4 Repulsive Field:

$$\begin{aligned}
U &= \frac{\alpha}{r}; \quad \alpha > 0 \\
U_{eff} &= \frac{\alpha}{r} + \frac{M^2}{2mr^2} \\
E &> 0
\end{aligned}$$

Only one possible motion, infinite, and the motion is a hyperbola. $r = \frac{p}{-1+e\cos\phi}$ so:

$$p = \frac{M^2}{m\alpha}; \qquad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}} > 1; \qquad r_{min} = \frac{p}{e-1}$$

7.5 Runge-Lenz Vector

The Runge-Lenz vector is a vector that describes the shape and orientation of the orbit of one body around another. (For $U = \frac{a}{r}$).

$$\begin{aligned}
\boldsymbol{D} &= \boldsymbol{v} \times \boldsymbol{M} + \alpha\frac{\boldsymbol{r}}{r} \quad \text{is constant} \\
\dot{\boldsymbol{D}} &= \dot{\boldsymbol{v}} \times \boldsymbol{M} + \alpha\frac{\boldsymbol{v}}{r} = \alpha\boldsymbol{r} \cdot \frac{\boldsymbol{v}\cdot\boldsymbol{r}}{r^3} \\
\frac{d}{dt}\frac{1}{\boldsymbol{r}} &= \frac{1}{\sqrt{\boldsymbol{r}\cdot\boldsymbol{r}}} = \frac{2\boldsymbol{v}\cdot\boldsymbol{r}}{2(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} = -\frac{\boldsymbol{v}\cdot\boldsymbol{r}}{r^3} \\
&\left.\begin{aligned} M &= m\boldsymbol{r} \times \boldsymbol{v} \\ m\dot{\boldsymbol{v}} &= \frac{\alpha\boldsymbol{r}}{r^3} \end{aligned}\right\} \text{equation of motion} \\
\dot{\boldsymbol{D}} &= \frac{\alpha\boldsymbol{r}}{r^3} \times (\boldsymbol{r} \times \boldsymbol{v}) + \alpha\frac{\boldsymbol{v}}{r} - \alpha\boldsymbol{r}\frac{\boldsymbol{v}\cdot\boldsymbol{r}}{r^3} = 0 \qquad \text{why?}
\end{aligned}$$

We now derive the formula $[a \times (b \times c)]_i$:

$$\begin{aligned}
\varepsilon_{ijk} a_j (b \times c)_k &= \varepsilon_{ijk}\varepsilon_{kmn} a_j b_m c_n \qquad \text{sum over repeated indices!} \\
&= \varepsilon_{ijk}\varepsilon_{mnk} a_j b_m c_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_j b_m c_n \\
&= a_n b_i c_n - a_m b_m c_i \\
&= b_i(\boldsymbol{a}\cdot\boldsymbol{c}) - c_i(\boldsymbol{a}\cdot\boldsymbol{b})
\end{aligned}$$

so:

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= b(\boldsymbol{a}\cdot\boldsymbol{c}) - c(\boldsymbol{a}\cdot\boldsymbol{b}) \\
\therefore \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{v}) &= r(\boldsymbol{r}\cdot\boldsymbol{v}) - \boldsymbol{v}r^2
\end{aligned}$$

Now,

$$\begin{aligned}
\boldsymbol{D}\cdot\boldsymbol{r} = Dr\cos\phi &= (\boldsymbol{v} \times \boldsymbol{m}) \cdot \boldsymbol{r} + \alpha\dot{\boldsymbol{r}} \\
&= \frac{(\boldsymbol{p} \times \boldsymbol{m}) \cdot \boldsymbol{r}}{m} + \alpha r = \frac{(\boldsymbol{r} \times \boldsymbol{p}) \cdot \boldsymbol{M}}{m} + \alpha r = \frac{M^2}{m} + \alpha r
\end{aligned}$$

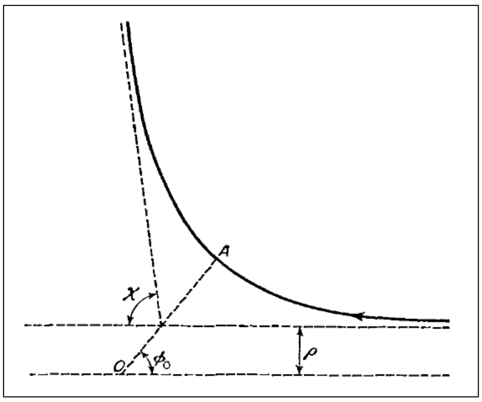
so:

$$1+\frac{M^2}{m\alpha}\frac{1}{r}=\frac{D}{\alpha}\cos\phi$$

now, since: $p=\frac{M^2}{m\alpha}$; $\quad e=\frac{D}{\alpha}$ we have:

$$\boxed{\frac{p}{r}=-1+e\cos\phi}$$

8 Scattering



$$\begin{aligned}\chi &= \text{ deflection angle.}\\ \chi &= \pi-2\phi_0 \text{ if the field is repulsive.}\\ \rho &= \text{ the impact parameter}\\ v_\infty &= \text{ the speed at infinity}\\ &\boxed{E=\frac{mv_\infty^2}{2}+U(\infty)}\end{aligned}$$

now:

$$\begin{aligned}\boldsymbol{m}=\boldsymbol{r}\times\boldsymbol{p};&\quad \boldsymbol{r}=\boldsymbol{r}_\parallel+\boldsymbol{\rho}\\ r\to\infty \implies&\quad r_\parallel\to\infty;\quad \boldsymbol{p}\to m\boldsymbol{v}_\infty\\ \boldsymbol{v}_\infty\parallel\boldsymbol{r}_\parallel \implies&\ \boldsymbol{r}_\parallel\times\boldsymbol{v}_\infty=0\\ \boldsymbol{M}=m\boldsymbol{\rho}\times\boldsymbol{v}_\infty \implies&\ \boxed{M=m\rho v_\infty}\end{aligned}$$

We now look to find out what ϕ_0 in terms of other variables is, using our old formula for ϕ:

$$\begin{aligned}
\phi &= \int \frac{(M/r^2)dr}{\sqrt{2m(E - U_{\text{eff}}(r))}} \\
U_{\text{eff}}(r) &= U(r) + \frac{M^2}{2mr^2} \\
r = \infty &\implies \phi = 0 \\
r = r_{min} &\implies \phi = \phi_0 \\
\phi_0 &= \int\limits_{r_{min}}^{\infty} \frac{(M/r^2)dr}{\sqrt{2m(E - U_{\text{eff}})}} \\
E - U_{\text{eff}} &= \underbrace{(E - U(\infty))}_{\frac{mv_\infty^2}{2}} - \underbrace{(U_{\text{eff}} - U(\infty))}_{U(r) - U(\infty) - \frac{M^2}{2mr^2}} \\
r = r_{min} &\implies E - U_{\text{eff}}(r_{min}) = 0 \\
M &= m\rho v_\infty \\
E &= \frac{mv_\infty^2}{2} \qquad (\text{if } U(\infty) = 0) \\
\phi_0 &= \int\limits_{r_{min}}^{\infty} \frac{(m\rho v_\infty/r^2)dr}{\sqrt{m^2 v_\infty^2 - 2mU - (m^2\rho^2 v_\infty^2/r^2)}} \\
&\boxed{\phi_0 = \int\limits_{r_{min}}^{\infty} \frac{(\rho/r^2)dr}{\sqrt{1 - (\rho^2/r^2) - (2U/mv_\infty^2)}}} \quad \text{Here, } U(\infty) = 0
\end{aligned}$$

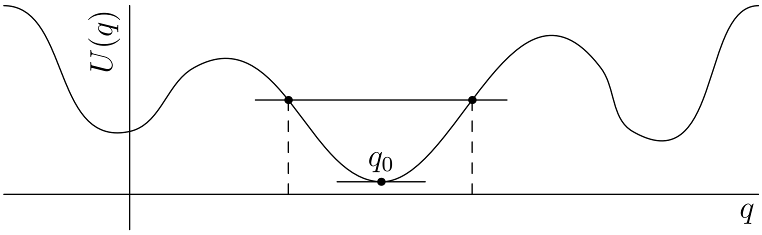
8.1 Coulomb Field

A Coulomb field is that where $U = \frac{\alpha}{r}$. We have then, setting $x = \frac{1}{r}$:

$$\phi_0 = \int\limits_{r_{min}}^{\infty} \frac{(\rho/r^2)dr}{\sqrt{1 - (\rho^2/r^2) - (\alpha/Er)}} = \int\limits_{0}^{x_{max}} \frac{\rho dx}{\sqrt{1 - \rho^2 x^2 - (\alpha x/E)}}$$

so:[10]

$$\begin{aligned}
\cos(\phi_0) &= \frac{(|\alpha|/mv_\infty^2\rho)}{\sqrt{1 + (\frac{\alpha}{mv_\infty^2})^2}} \\
\cot \chi/2 &= \frac{\rho m v_\infty^2}{|\alpha|}
\end{aligned}$$



9 Oscillations

9.1 Oscillations in one dimension

The general Lagrangian for one-dimension is $L = \frac{1}{2}a(q)\dot{q}^2 - U(q)$. Suppose we have a stable equilibrium point q_0, as in the diagram above. Suppose then we look at a small displacement:

$$q = q_0 + x \implies q - q_0 = x$$
$$U(q) - U(q_0) = U'(q_0)x + \frac{1}{2}U''(q_0)x^2 + \dots$$

However, $U'(q_0) = 0$ since it is a minimum, so we start our expansion from $\frac{1}{2}U''(q_0)x^2 = \frac{1}{2}kx^2$. Note that $a(q)$ is our generalised mass, and we have $a(q) = a(q_0) + a'(q_0)x + \ldots \approxeq m$, $a(q_0) = m$. The point is that for the approximation, we replace L with the Lagrangian:

$$\boxed{L \to L = \frac{m}{2}\dot{x}^2 - \frac{1}{2}kx^2; \quad k > 0} \qquad \leftarrow \text{one dimensional oscillator}$$

Note that we must have $m > 0$, otherwise we'd never have a minimum. Our equation of motion is

$$m\ddot{x} + kx = 0 \quad \text{or}$$
$$\boxed{\ddot{x} + \omega^2 x = 0} \qquad \boxed{\omega^2 = \frac{k}{m} = \text{freq. of oscillator}}$$

The general solution is given by:

$$x = c_1 \cos\omega t + c_2 \sin\omega t$$
$$= a\cos\underbrace{(\omega t + \alpha)}_{\text{phase}}$$
$$a = \sqrt{c_1^2 + c_2^2} = \text{ amplitude}; \quad \tan\alpha = -\frac{c_2}{c_1}$$

α is the initial phase

$$E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 = \frac{k}{2}a^2 = \frac{1}{2}m\omega^2 a^2$$

---

[10] This is tedious to show...integrate by completing the square then find what x_{max} is

If we switch to complex ‘coordinates’:

$$\begin{aligned}
\xi &= \dot{x} + i\omega x \\
i\omega\xi &= i\omega\dot{x} - \omega^2 x \\
\dot{\xi} &= \ddot{x} + i\omega\dot{x}
\end{aligned}$$

so:

$$\dot{\xi} - i\omega\xi = m\ddot{x} + kx = 0$$

This is a 1st order differential equation, much easier to solve than a second order one!

$$\boxed{\xi = Ce^{i\omega t}; \quad C \in \mathbb{C}}$$

$$\begin{aligned}
x \in \mathbb{R} &\implies \Re(\xi) = \dot{x}; \quad \Im(\xi) = \omega x \\
x &= \frac{1}{\omega}\Im(Ce^{i\omega t}); \quad \text{let } C = i\omega a e^{i\alpha} \text{ so:} \\
&= a\Im(ie^{i(\omega t+\alpha)}) \\
&= a\Im(i(\cos{(\omega t + \alpha)} - \sin{(\omega t + \alpha)})) \\
&= a\cos{(\omega t + \alpha)} \quad \text{(as before)}
\end{aligned}$$

then, if we let $A = ae^{i\alpha}$ we have:

$$\boxed{x = \Re(Ae^{i\omega t}); \quad |A| = a; \quad \arg{(A)} = \alpha}$$

9.2 Forced Oscillations

For forced oscillations, we modify our Lagrangian by adding a term for an external field, giving us:

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 - U_e(x,t)$$

So we do an expansion again:

$$U_e(x,t) \approxeq \underbrace{\cancel{U_e(0,t)}}_{=\frac{df}{dt}} + \underbrace{\frac{\partial U_e(0,t)}{\partial x}}_{=-F(x)} x + \dots$$

where the first term can be cancelled out since it is a function of time only and can therefore be omitted from the Lagrangian as being the total time derivative of another function of time. So our new Lagrangian is:

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 + F(t)x$$

giving us the new equation of motion:

$$\ddot{x} + \omega^2 x = \frac{1}{m}F(t)$$

This is a linear inhomogeneous differential equation. The general solution is $x(t) = x_0(t) + x_1(t)$. We now solve the equation:

$$\begin{aligned} \xi &= \dot{x} + i\omega x \\ \dot{\xi} - i\omega\xi &= \frac{1}{m}F(t) \end{aligned}$$

we make the ansatz[11]$\xi = C(t)e^{i\omega t}$ so:

$$\dot{\xi} = \dot{C}e^{i\omega t} + i\omega = \dot{C}e^{i\omega t} + i\omega t$$

so:

$$\begin{aligned} \dot{C}e^{i\omega t} &= \frac{F(m)}{m} \\ \dot{C} &= \frac{1}{m}F(t)e^{-i\omega t} \\ C(t) &= \int\limits_0^t \frac{1}{m}F(\tau)e^{-i\omega\tau}d\tau + C(0) \\ \xi &= e^{i\omega t}\left(\int\limits_0^t \frac{1}{m}F(\tau)e^{-i\omega\tau}d\tau + \xi_0\right) \quad \text{where } C(0) = \xi_0 \end{aligned}$$

then, since $\xi = \dot{x} + i\omega x$:

$$\boxed{x(t) = \frac{1}{\omega}\Im(\xi)} \quad \leftarrow \text{a particular solution could be } \xi(t) \text{ with } \xi_0 = 0$$

9.3 Energy of an oscillating system

The energy of an oscillating system is not conserved, since the system gains energy from the source of the external field. We wish to find the total energy transmitted to the system. Let $x(-\infty) = 0$. Then:

$$\begin{aligned} E &= \frac{1}{2}m(\dot{x} + \omega^2x^2) = \frac{1}{2}|\xi|^2 \\ E(+\infty) &= \frac{1}{2}m\left|\int\limits_{-\infty}^{+\infty} \frac{1}{m}F(\tau)e^{-i\omega\tau}d\tau\right|^2 \\ &= \frac{m}{2}|\tilde{F}(\omega)|^2 \quad \text{(where } \tilde{F} \text{ is the Fourier component}^{12}\text{of } F(t)\text{)} \end{aligned}$$

Thus the energy imparted to the system is determined by the squared modulus of Fourier component of the force $F(t)$ whose frequency is the intrinsic frequency of the system.

---

[11] An ansatz is a trial solution based on an educated guess.

[12] How do we arrive at this result? Note that Fourier analysis tells us we can write $f(x)$ as:

$$f(x) = \int\limits_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk \quad \text{where} \quad \tilde{f}(k) = \int\limits_{-\infty}^{+\infty} \frac{1}{2\pi}f(x)e^{-ikx}dk$$

9.4 Resonance

We now look at resonance of an oscillating system. Suppose the driving force is given by $F(t) = \frac{f}{m}\cos(\gamma t + \beta)$. Then:

$$\ddot{x} + \omega^2 x = \frac{f}{m}\cos(\gamma t + \beta) \quad \text{where} \quad x = \underbrace{x_0(t)}_{\text{homogeneous sol.}} + \underbrace{x_1(t)}_{\text{any particular sol.}}$$

we know:

$$x_0(t) = a\cos(\omega t + \alpha)$$
$$x_1(t) = ? \quad \text{2nd derivative of cos is itself!}$$

We use the ansatz $x_1 = b\cos(\gamma t + \beta)$ to transform $\ddot{x} + \omega^2 x = \frac{f}{m}\cos(\gamma t + \beta)$ into:

$$(-b\gamma^2 + b\omega^2)\cos(\gamma t + \beta) = \frac{f}{m}\cos(\gamma t + \beta) \quad \leftarrow \text{time dependence disappears}$$

Thus, we arrive at $b = \dfrac{f}{m(\omega^2 - \gamma^2)}$ and we arrive at

$$\boxed{x = \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta) \quad (\text{if } \omega \neq \gamma)}$$

If $\gamma = \omega$ we have *resonance*. We wish to analyse this situation. Consider the limit $\gamma \to \omega$. Assume $\gamma = \omega + \varepsilon; \varepsilon \ll \omega$. Then:

$$\omega^2 - \gamma^2 = (\omega - \gamma)(\omega + \gamma) \approxeq -2\omega\varepsilon$$

Take as a particular solution:

$$x_1 = a_1\cos(\omega t + \alpha_1) - \frac{f}{2m\omega\varepsilon} cos(\omega t + \beta + \varepsilon t)$$

we use $\cos(A + B) = \cos A\cos B - \sin A\sin B$ and the small angle approximation formulae here:

$$= a_1\cos(\omega t + \alpha_1) - \frac{f}{2m\omega\varepsilon}\cos(\omega t + \beta) + \frac{f}{2m\omega}\cdot t\sin(\omega t + \beta)$$

We want x_1 to have a finite limit as $\varepsilon \to 0$, so we set $a_1(\varepsilon) = \frac{f}{2m\omega\varepsilon}$

$$= \frac{f}{2m\omega\varepsilon}\left[\cos(\omega t + \alpha_1) - \cos(\omega t + \beta)\right] + \frac{f}{2m\omega}\cdot t\sin(\omega t + \beta)$$

setting a_1 to β:

$$\boxed{x_1 = \frac{f}{2m\omega} t\sin(\omega t + \beta)} \quad \leftarrow \text{The particular solution we're looking for}$$

Relabel $f(x)$ as $F(t)$ and $\tilde{f}(k)$ as $\tilde{F}(\omega)$ and we get the result.

We see that amplitudes increase linearly in time (until the oscillations no longer count as small, at which point the theory breaks down.) We now look at what happens with small oscillations near resonance.

$$\begin{aligned} \gamma &= \omega + \varepsilon; \quad \varepsilon \ll \omega \\ x &= a\cos(\omega t + \alpha) + b\cos(\omega t + \beta) \end{aligned}$$

substituting in $x = Ae^{i\omega t} + Be^{i\gamma t} = (A + Be^{i\varepsilon t})e^{i\omega t}$ we get:

$$\begin{aligned} A &= ae^{i\alpha}; \quad B = be^{i\beta} \\ x &= (A + B) \\ C(t) &= A + Be^{i\epsilon t} \\ x(t) &= C(t)e^{i\omega t} \end{aligned}$$

We want to find the max/min of the complex amplitude $C(t) = C_0(t)e^{i\varepsilon t}$

$$\begin{aligned} |C|^2 &= (A + Be^{i\varepsilon t})(\bar{A} + \bar{B}e^{-i\varepsilon t}) \\ &= |A|^2 + |B|^2 + A\bar{B}e^{-i\varepsilon t} + \bar{A}Be^{i\varepsilon t} \\ &= a^2 + b^2 + ab\left(e^{i(\alpha-\beta-\varepsilon t)} + e^{-i(\alpha-\beta-\epsilon t)}\right) \\ &= a^2 + b^2 + 2ab\cos(\varepsilon t + \beta - \alpha) \end{aligned}$$

therefore, $(a-b)^2 \leq |C|^2 \leq (a+b)^2$, or

$$\boxed{|a-b| \leq |C| \leq |a+b|}$$

which describes the phenomenon called *beats*. Now:

$$x(t) = \Re(C(t)e^{i\omega t})$$

Let $\alpha = \beta = 0$

$$C(t) = a + be^{i\varepsilon t} = a + b\cos(\varepsilon t) + \underbrace{\cancel{ib\sin(\varepsilon t)}}_{0 \text{ since } \varepsilon \ll 1}$$

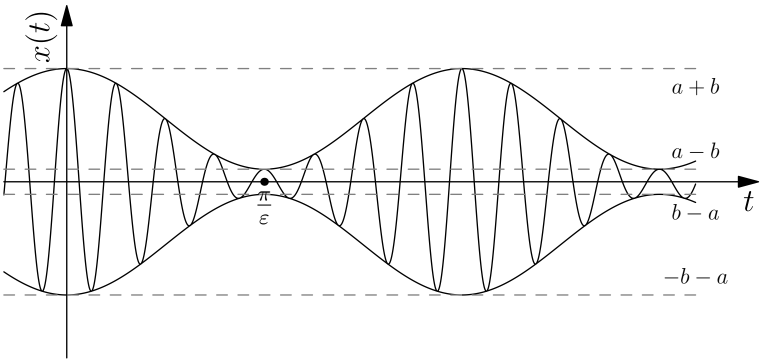
$$\boxed{x(t) \approxeq (a + b\cos(\varepsilon t))\cos(\omega t) = C(t)\cos(\omega t)}$$

thus the behaviour should be obvious, we have our original wave, the amplitude of which is modulated with frequency ε between $|a-b|$ and $|a+b|$.

9.5 Oscillation of systems with many degrees of freedom

We now look at systems which have many degrees of freedom in which they can oscillate.

$$L = \frac{1}{2}g_{jn}(q)\dot{q}_j\dot{q}_n - U(q_1, q_2, \ldots, q_n); \qquad q_n : n = 1, \ldots, s$$



We take $q_n^{(0)}$ as our equilibrium point, and define $x_n = q_n - q_n^{(0)}$, and expand about $U(q) = U(q^{(0)} + x)$:

$$U(q) = U(q^{(0)} + x) \approxeq U(q^{(0)}) + \partial j U(q^{(0)}) x_j + \frac{1}{2} \partial j \partial k U(q^{(0)}) x_j x_k + O(x^3)$$

$q^{(0)}$ is equilibrium implies $U(q^{(0)})$ is a local minimum of $U(q)$ which in turn implies it is zero, as in the single dimensional case. Then, taking the minimum of the potential energy to be zero like we did before, $U(q^{(0)}) = 0$, so we have:

$$U(q) = \frac{1}{2} K_{jn} x_j x_n = \frac{1}{2} \partial j \partial n U(q^{(0)}) x_j x_n$$

where K is:

$$K = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1S} \\ K_{21} & K_{22} & \cdots & K_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ K_{S1} & K_{S2} & \cdots & K_{SS} \end{pmatrix} ; \quad K^T = K \quad \text{(K is positive definite}^{13}\text{)}$$

Note that we have $K^T = K$ since the coefficients K_{jn} and K_{nj} multiply the same quantity $x_j x_n$ and can therefore be considered equal. Since K is positive definite, we can diagonalise it under some change of basis $K \to A^T K A = X$ for some orthogonal[14]matrix A. Thus we have:

$$X = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_S \end{pmatrix} ; \quad K_n > 0 \; \forall \; n = 1, \ldots, s$$

Thus for the two dimensional case, our potential becomes:

$$U = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2$$

---

[13] K is positive definite since the potential must increase in all directions from the equilibrium point.

[14] Orthogonal matrix: $A^T A = I$.

We can introduce the mass matrix $m_{jn} = g_{jn}(q^{(0)})$, $m_{jn} = m_{nj}$ so that we obtain the kinetic energy as a positive definite quadratic form, and we arrive at our general Lagrangian for small oscillations:

$$\boxed{L = \frac{1}{2} m_{jn} \dot{x}_j \dot{x}_n - \frac{1}{2} K_{ij} x_i x_j}$$

Our equations of motion are thus: $m_{jn}\ddot{x}_n + K_{jn}x_n = 0$ which form a set of linear homogeneous second order differential equations. Using the ansatz $x_n = A_n e^{i\omega t}$; $n = 1, \ldots, s$; $A_n, \omega \in \mathbb{C}$, whence $\ddot{x}_n = -\omega^2 A_n e^{i\omega t}$, we arrive at $(-\omega^2 m_{jn} + k_{jn})A_n = 0$. Now, let $\Omega_{jn} = k_{jn} - \omega^2 m_{jn}$. Note that $\Omega_{jn}A_n = 0 \equiv \Omega \cdot A = 0$. Now:

$$\begin{aligned} \text{Let: } & \det \Omega \neq 0 \implies \exists\, \Omega^{-1} \text{ such that } \Omega^{-1}\Omega = I \\ & \Omega \cdot A = 0 \implies \Omega^{-1}\Omega \cdot A = \Omega^{-1} \cdot 0 \implies I \cdot A = 0 \implies A = 0 \\ \therefore\ & \det \Omega = 0 \quad \text{for non-trivial solutions} \end{aligned}$$

Thus we have our *characteristic equation* $|k_{jn}| - \omega^2 m_{jn}| = 0$ which is of degree S in ω^2. The quantities ω_α corresponding to the roots $\omega_\alpha^2; \alpha = 1, 2 \ldots, S$ are then the *characteristic frequencies* or *eigenfrequencies* of the system. Note too that ω^2 must be real and positive, since otherwise we would have complex ω in $x = A_n e^{i\omega t}$, which lead to an exponentially increasing or decaying factor in $\dot{x}$, which in turn would lead to a time variation of the total energy $E = U + T$, which would therefore not be conserved.

Now, let's look at the case where all ω_α^2 are different. In that case the rank of $(k_{jn} - \omega^2 m_{jn})$ is $S - 1$. Then, there exists one solution for each ω_α. We now wish to find, for each ω_α, the S corresponding coefficients $A_j^{(\alpha)}$, which turn out[15]to be proportional to the minors $\Delta_{j\alpha}$ of the determinant $(k_{jn} - \omega^2 m_{jn}$. Each of these $A_j^{(\alpha)}$ are particular solutions for some frequency ω_α. Thus our general solution is a linear combination :

$$x_j = \Re \sum_{\alpha=1}^{S} \Delta_{j\alpha} \underbrace{C_\alpha}_{\text{complex constant}} e^{i\omega_\alpha t} = \sum_{\alpha=1}^{S} \Delta_{k\alpha}\Theta_\alpha$$

where:

$$\Theta_\alpha = \Re(C_\alpha e^{i\omega_\alpha t})$$

so:

$$\ddot{\Theta}_\alpha + \omega_\alpha^2 \Theta_\alpha = 0 \quad \text{(no summation over } \alpha)$$

then Θ_α become new generalised coordinates called *normal coordinates.* Importantly when expressed like this we see the equations of motion become S independent equations. We can write:

$$x_j = \sum_{\alpha=1}^{S} \Delta_{j\alpha}\Theta_\alpha \quad \leftarrow \text{solve for } \Theta_\alpha$$

---

[15] I have no idea where this comes from, it is written on page 67 of L&L. If you understand this, please email me.

It is evident that Lagrangian expressed in terms fo normal coordinates is a sum of expressions each of which corresponds to an α, i.e.

$$L = \sum_{\alpha} \frac{1}{2} m_{\alpha} (\dot{\Theta}_{\alpha}^2 - \omega_{\alpha}^2 \Theta_{\alpha}^2)$$

or, defining new normal coordinates $Q_{\alpha} = \sqrt{m_{\alpha}}\Theta_{\alpha}$:

$$L = \frac{1}{2} \sum_{\alpha} (\dot{Q}_{\alpha}^2 - \omega_{\alpha}^2 Q_{\alpha}^2)$$

We can summarise then: We have

$$\boxed{L = \frac{1}{2} m_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} x_i x_j} = \boxed{\frac{1}{2} \dot{X}^{\intercal} M \dot{X} - \frac{1}{2} X^{\intercal} K X}$$

1. Diagonalise M: $M = A\mu A^{\intercal};\ \ \mu = \text{diag}(\mu_1, \mu_2, \ldots, \mu_S);\ \ AA^{\intercal} = A^{\intercal}A = I$
   So $L = \frac{1}{2}\dot{X}^{\intercal} A\mu A^{\intercal}\dot{X} - \frac{1}{2}\dot{Y}^{\intercal}Y - \frac{1}{2}Y^{\intercal}A^{\intercal}KA$ and $Y = A^{\intercal}X; X = AY$.

2. Rescale Y so that $\dot{Y}^{\intercal}\sqrt{\mu}^2\dot{Y} = \dot{Z}^{\intercal}\dot{Z} = \sum\limits_{i=1}^{S} \dot{Z}_i^2$

   So $Z = \sqrt{\mu}Y;\ \ Y = \frac{1}{\sqrt{\mu}}Z\ \ \sqrt{\mu} = \text{diag}(\sqrt{\mu}_1, \sqrt{\mu}_2, \ldots, \sqrt{\mu}_S)$

   Then $L = \frac{1}{2}\dot{Z}^{\intercal}\dot{Z} - \frac{1}{2}Z^{\intercal}\underbrace{\frac{1}{\sqrt{\mu}}A^{\intercal}KA\frac{1}{\sqrt{\mu}}}_{\tilde{K}}Z;\ \ \tilde{K} = \frac{1}{\sqrt{\mu}}A^{\intercal}KA\frac{1}{\sqrt{\mu}};\ \ \tilde{K} = \tilde{K}^{\intercal}$

3. Diagonalise $\tilde{K} : \tilde{K} = BXB^{\intercal};\ \ X = \text{diag}(X_1, X_2, \ldots, X_S)$
   $L = \frac{1}{2}\dot{Z}^{\intercal}Z - \frac{1}{2}Z^{\intercal}BX\underbrace{B^{T}Z}_{Q} = \frac{1}{2}\dot{Q}^{\intercal}\dot{Q} - \frac{1}{2}Q^{\intercal}XQ$

   $= \frac{1}{2}\sum\limits_{i=1}^{S}(\dot{q}_i^2 - X_i q_i^2);\ \ \omega_i^2 = X_i$