The Pendulum

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February 19, 2012

Abstract

The motion of a simple pendulum was modelled using the linear approximation of the equations of motion, valid for small angles, and using numerical methods to solve the nonlinear equations of motion, namely the trapezoid rule and fourth order Runge-Kutta method. The trajectories, suitably graphed with gnuplot, were compared, and finally the Runge-Kutta method was used to simulate a damped driven pendulum at various amplitudes of the driving force, with the phase portrait plotted again in gnuplot to show dynamical features such as period doubling and chaotic motion.

Introduction and Theory

From Newton's Second Law, it follows that the equation of motion for a simple pendulum in a plane is

$$\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2} = -g\sin(\theta)$$

where L is the length of the string and $g\sin\theta$ is the downward acceleration due to gravity.

This is a non-linear equation, but can be simplified using the small angle approximation $\sin(\theta) \approx \theta$, giving the second order differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

which can be solved to give the familiar equation for the harmonic oscillator

$$\theta = A\sin(\beta t + \phi), \beta = \sqrt{\frac{g}{L}}$$

where A is the amplitude and ϕ is the constant phase offset.

Solving Second Order Differential Equations Numerically

A second order differential equation can be solved numerically, both in the linear and non-linear cases by transforming it into two first-order equations, as shown here for the pendulum equation:

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\beta^2 \sin(\theta)$$

Damped Driven Oscillator

The model of the simple pendulum can be expanded to take into account damping and driving forces. In this case, the equation of motion is given by

$$\frac{d^2\theta}{dt^2} + k\omega + \beta^2 \sin(\theta) = A\cos(\Omega t)$$

Derivation of the formula for G

In this experiment, aside from the various masses and distances between objects, as described below, the main measurements to be made were on the period of oscillation, T, of the torsion balance, and the positions of equilibrium, in terms of the deflection angles $\alpha_1 and \alpha_2$, about which the balance oscillated. From these then, can the gravitational constant G be measured. The argument proceeds thus:

When the pendulum is equilibrium in position 1, the gravitational torque, τ_1 due to the attraction between the pairs of spheres is equal to the restoring torque, $k\alpha_1$, of the wire due to the torsion. Thus, $k\alpha_1 = \tau_1$. Similarly, $k\alpha_2 = \tau_2$.

The gravitational attraction between a large sphere and the smaller one is, by Newton's law of gravitation, $F = GMm/b^2$ where b is the separation between the two spheres. Therefore the torque due to both pairs of spheres is

$$\tau = 2\frac{GmM}{b^2}d$$

where d is the distance from the centre of each small sphere to the axis of rotation.

Thus summing the expressions for $k\alpha_1$ and $k\alpha_2$, we have

$$k \cdot (\alpha_1 + \alpha_2) = 4G\frac{mM}{b^2}d$$

The general formula for the period of a harmonic oscillator is

$$T = 2\pi\sqrt{\frac{\text{Inertia}}{\text{Restoring Force}}}$$

thus, for our torsion pendulum we have

$$T^2 = 4\pi^2\frac{I}{k}$$

now the moment of inertia for our pendulum can be modelled as two point masses each a distance d from the axis of our rotation, $I = 2d^2m$, giving us $T^2 = 8\pi^2 d^2 m$, or

$$k = \frac{8\pi^2 d^2 m}{T^2}$$

Oscillation of the balance

When the lead sphere is in position 1, it performs oscillations, in the plane of all the masses, about a particular equilibrium position α_1. While in theory, the experiment can be left long enough for the oscillations to effectively decay completely, in reality this isn't possible due to the eventuality of outside forces affecting the experiment.

Any force affecting the motion of the masses will result in an unwanted oscillation of the torsion balance, in any of several modes of oscillation, leading to For this reason it was necessary to fit the relevant parts of the actual data to a general damped harmonic oscillator function, $y = Ae^{-\gamma t} \cos(\omega t - \alpha)$

Experimental Method

The lead spheres were placed in position 1, so that the small lead balls oscillate about equilibrium position α_1. The data for the oscillations from the infrared detector were recorded for several full periods of oscillations of the pendulum, i.e. over a time frame from about half an hour to an hour.

Then, the lead spheres were rotated slowly to position 2, and again data from the detector were recorded.

Finally, the equilibrium points and the period of oscillation were measured from the data.

Results and Analysis

Discussion and Conclusion

The general theme to the entire experiment was the difficulty in getting a set of usable results without the apparatus being disturbed. The entire run for the first week did not result in any usable results, while the results from the second week were only usable thanks to the help from Prof. Stamenov with using nonlinear regression to fit a curve to a subset of data which hadn't been contaminated (detected by looking for unexpected inflections in the curve).

After fitting the curve to the data, it appeared as if the actual data was going to be quite accurate, however the final result was disappointingly far off from the accepted value.