The Pendulum

Andrew Mark Allen - 05370299

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Abstract

The motion of a simple pendulum was modelled using the linear approximation of the equations of motion, valid for small angles, and using numerical methods to solve the nonlinear equations of motion, namely the trapezoid rule and fourth order Runge-Kutta method. The trajectories, suitably graphed with gnuplot, were compared, and finally the Runge-Kutta method was used to simulate a damped driven pendulum at various amplitudes of the driving force, with the phase portrait plotted again in gnuplot to show dynamical features such as period doubling and chaotic motion.

Introduction and Theory

The Simple Pendulum

From Newton's Second Law, it follows that the equation of motion for a simple pendulum in a plane is

$$\frac{d^2 s}{dt^2} = L\frac{d^2\theta}{dt^2} = -g\sin(\theta)$$

where L is the length of the string and $g\sin\theta$ is the downward acceleration due to gravity.

This is a non-linear equation, but can be simplified using the small angle approximation $\sin(\theta) \approx \theta$, giving the second order differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

which can be solved to give the familiar equation for the harmonic oscillator

$$\theta = A\sin(\beta t + \phi), \beta = \sqrt{\frac{g}{L}}$$

where A is the amplitude and ϕ is the constant phase offset.

Solving Second Order Differential Equations Numerically

A second order differential equation can be solved numerically, both in the linear and non-linear cases by transforming it into two first-order equations, as shown here for the pendulum equation:

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\beta^2 \sin(\theta)$$

Damped Driven Oscillator

The model of the simple pendulum can be expanded to take into account damping and driving forces. In this case, the equation of motion is given by

$$\frac{d^2\theta}{dt^2} + k\omega + \beta^2 \sin(\theta) = A\cos(\Omega t)$$

where k is a damping coefficient, A is the amplitude of the periodic driving force, and Ω is its angular frequency.

Simple Euler Method

The simple Euler Method is one of the simpler ways for finding numerical solutions for the time evolution of the system. It involves taking an initial value of (in this case) θ and ω and using a Taylor expansion to find the values after a time increment. Discarding terms of order 2 or higher gives us a final value (for θ) of

$$\theta_{n+1} = \theta_n + \omega_n \Delta t$$

Errors per step in the simple Euler method are of the order Δt^2, and the total error is of the order Δt.

Trapezoid Rule

The trapezoid rule is a better method, which involves approximating the area under the curve as a trapezoid. This gives an improvement that is accurate to second order terms.

Runge-Kutta Method

The fourth order Runge Kutta is a linear approximation where θ_{n+1} is given by

$$\theta_{n+1} = \theta_n + \frac{h}{6}(k_{1a} + 2k_{2a} + 2k_{3a} + k_{4a})$$

Where the k's are values of the slope of the function at the beginning of the interval, midpoint of the interval and at the end of the interval. It is called the fourth order Runge-Kutta method because errors are of the order Δt^4 and is therefore much better than the simple Euler Method.

Experimental Method

A program using the trapezoid rule was used to simulate the simple pendulum system. It was then modified to describe the non-linear pendulum, and to use the Runge-Kutta method. In all cases four different test cases were used, consisting of starting conditions of $\theta = (0, 0.2, 1.0, 3.124)$ and $\omega = (1, 0, 0, 0)$. In all cases the results were then plotted using gnuplot.

Results and Analysis

Linear Approximation

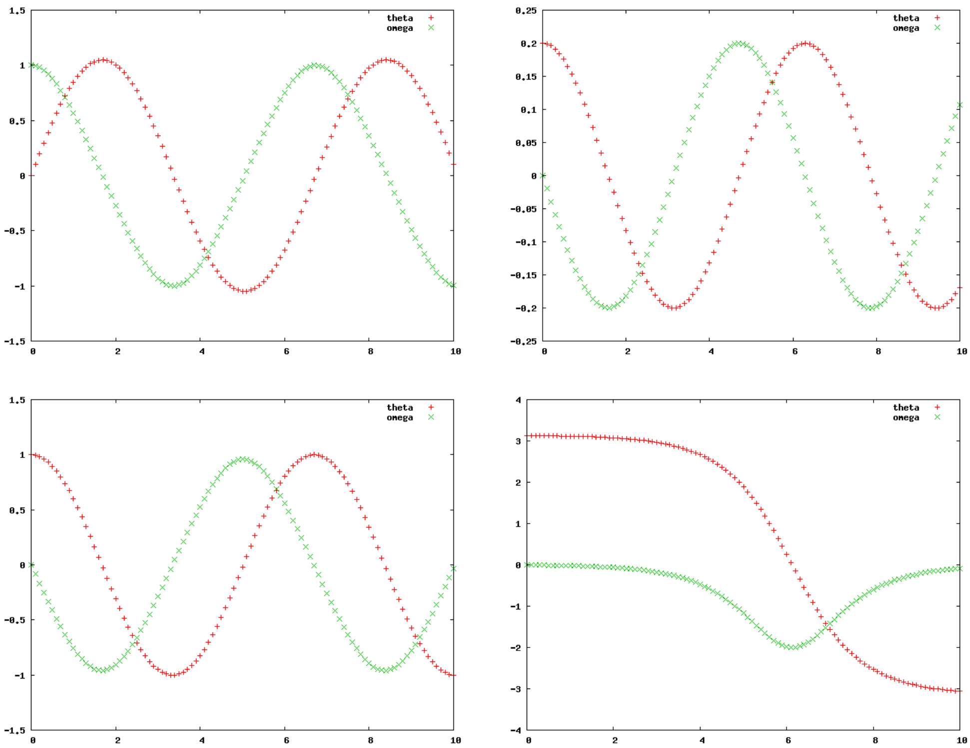

Figure 1: Linear approximation

This is the linear approximation, or small-angle approximation, which gives sufficiently good values for small starting angles, but gives the same (and therefore less realistic) perfectly sinusoidal results for higher starting angles.

Non-linear Approximation

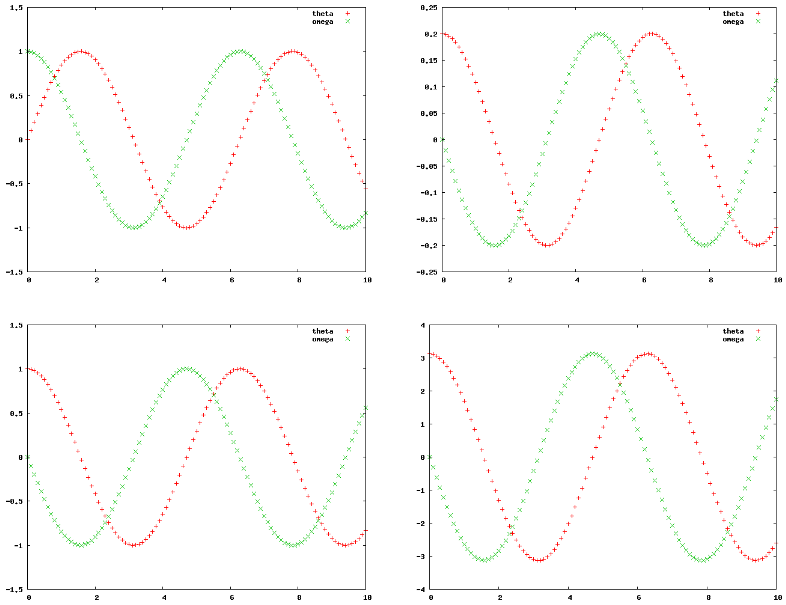

Figure 2: Non-linear approximation

The non-linear approximation shows clearly distinct behaviour for small starting angles and large ones.

Comparison of Linear and Non-Linear Approximation

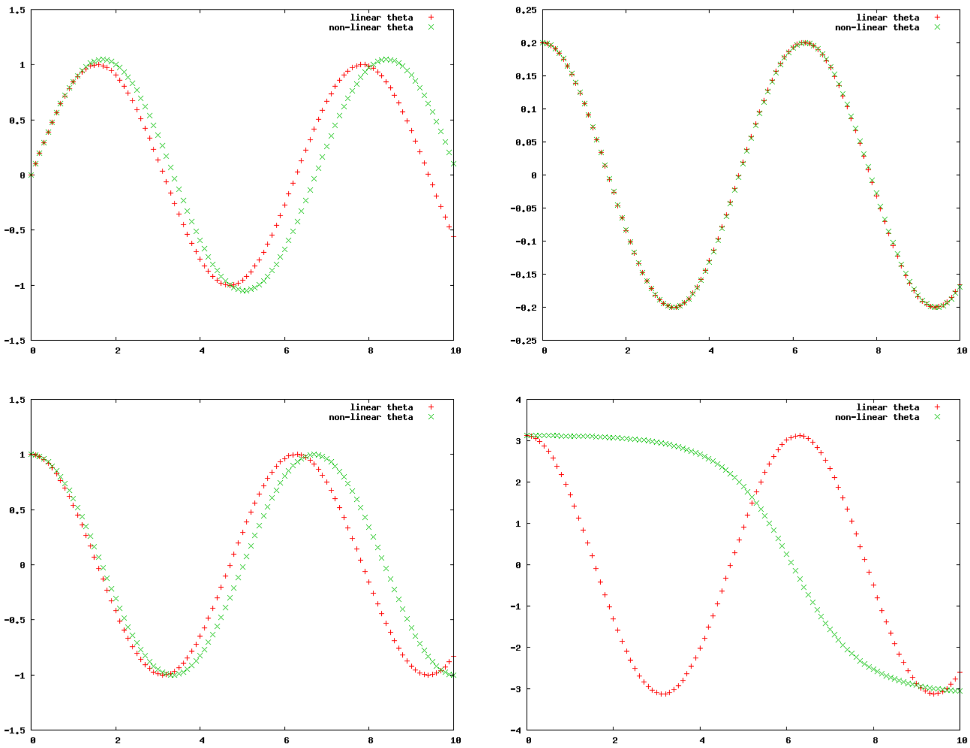

Figure 3: Comparison of linear and non-linear behaviour

It can be seen in Figure: 3 that the at higher starting angles, the linear behaviour becomes increasingly wrong compared to the more realistic behaviour. The linear behaviour remains perfectly sinusoidal in all cases, as this is the central assumption of the approximation. In comparison, the non-linear behaviour gives a far more accurate result in the last picture to what we expect would happen when a simple pendulum (with a rigid 'string') is released from near vertical: It takes a relatively long time to accelerate from its starting point of near-equilibrium, and slowly decelerates to near the same point on the opposite side.

Runge-Kutta approximation

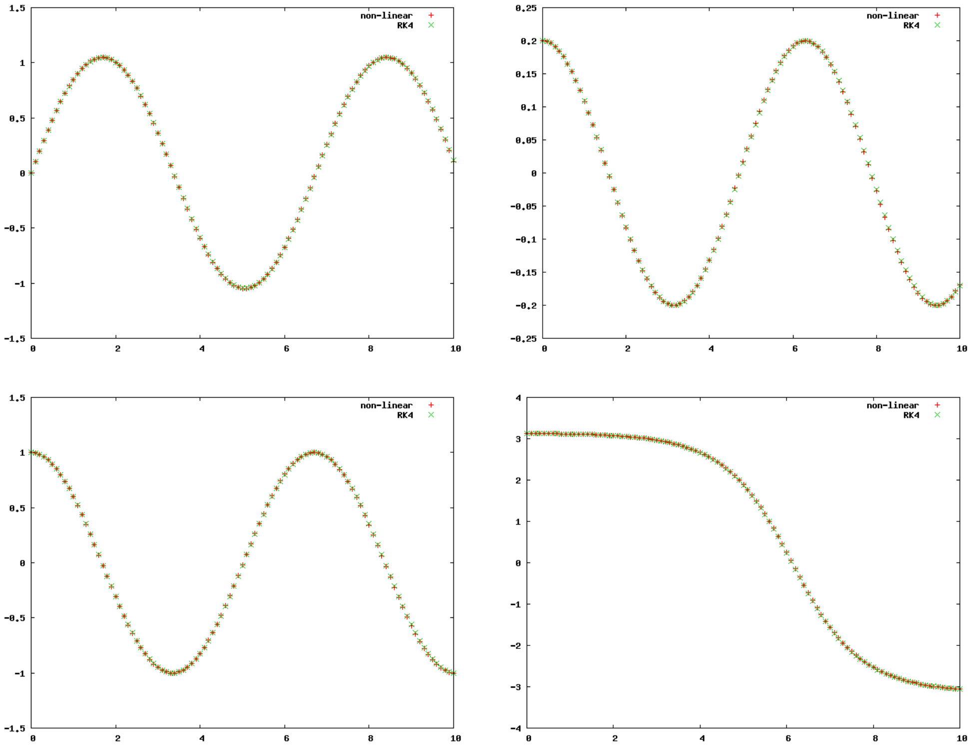

Figure 4: Runge-Kutta versus non-linear approximation

This comparison of the Runge-Kutta fourth order approximation to the non-linear approximation shows that there's very little difference in the values of theta throughout the simulation, and that therefore the

Damped Oscillator

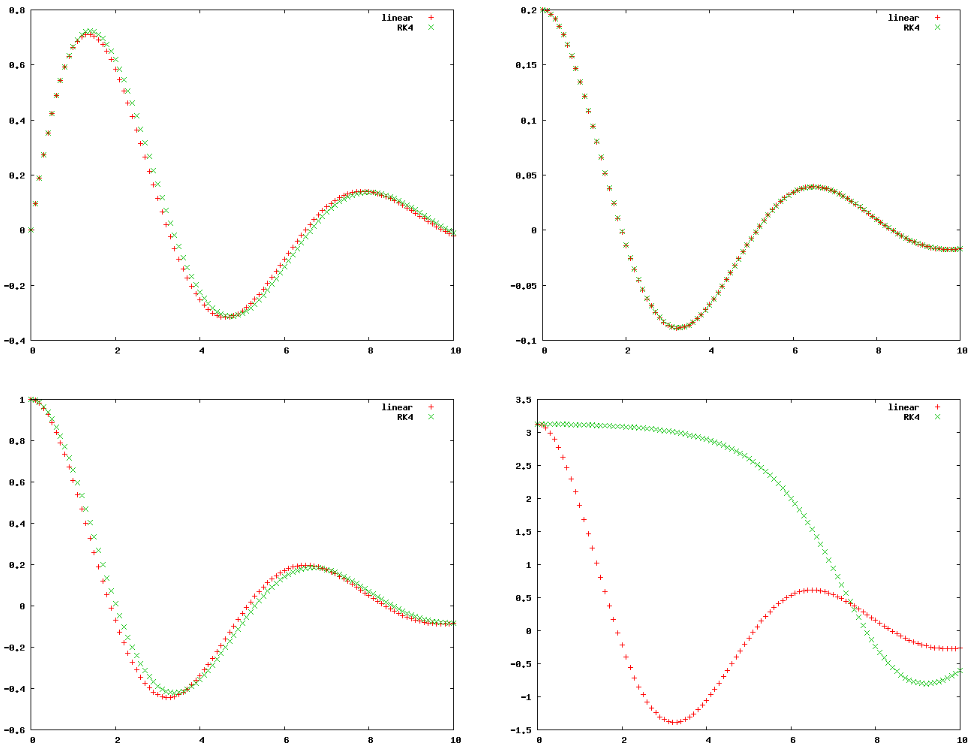

Figure 5: Comparison of linear and non-linear simulation for damped pendulum

The comparison for the damped pendulum show that again, for small starting angles the linear approximation is sufficient, but as the starting angle increases, the linear approximation becomes less and less appropriate, although both simulations converge to the same final value (static equilibrium).

Damped Driven Oscillator

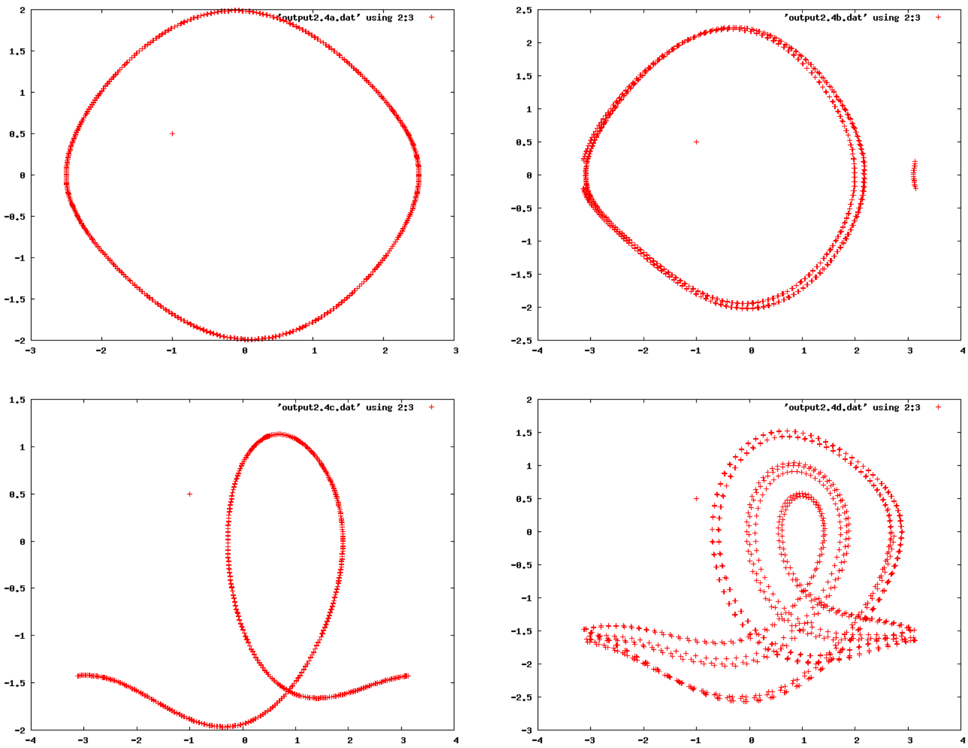

Figure 6: Damped driven pendulum behaviour

The final results pertain to the damped driven oscillator. These diagrams are phase portraits that show the trajectories of the pendulum, in the form of θ against ω. The driving of the amplitude is increased across these diagrams, and when the amplitude gets sufficiently large, the trajectory of the pendulum becomes complex and chaotic, with period doubling and chaotic behaviour appearing.

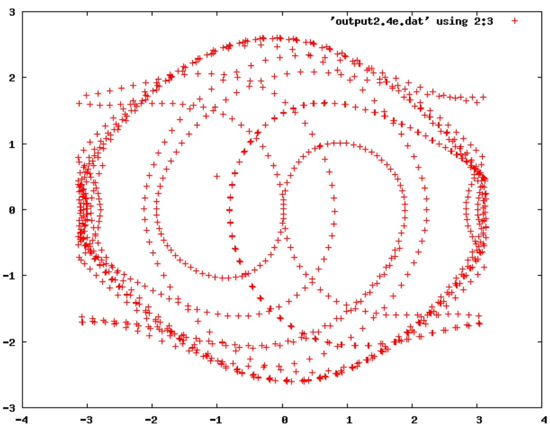

Figure 7: Chaotic behaviour for damped driven pendulum

This diagram shows a plot over a longer time period than the previous diagrams, showing clearly the chaotic behaviour of the pendulum for large amplitudes of driving force.

Discussion and Conclusion

From the plots, it can be seen that the small angle approximation is, as its name implies, only suitable for relatively small displacements of the pendulum (less than 10 deg or so.) After that more suitable models are needed. The approximation was suitable for simulated the damped pendulum at small angles however, since the values eventually converged to 0 in all cases as time went on.

The Runge-Kutta method was far superior at higher displacement angles, and allowed the equation of motion for a damped driven oscillator to be solved. With this method, the motion of the oscillator could be analysed after being plotted on a phase space graph. As the amplitude of the driving force increased, the complexity of the motion could be seen to increase, as first period doubling appeared in the motion, and finally fully chaotic motion was observed.