

Theorems of Complex Analysis

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1 Introduction

List of theorems.

1.1 Test

2 Cauchy's Theorem

Cauchy's integral theorem states that for a closed path, the path integral of an analytic function over the path will be zero. We prove this theorem first for a triangular path and then for a star-shaped set.

Lemma 1. *Let $F_1 \supset F_2 \supset F_3 \supset \dots$ be a decreasing sequence of non-empty closed sets. If $\lim_{n \rightarrow \infty} \text{diam}(F_n) = 0$ then $\bigcap_{n=1}^{\infty} F_n$ is non-empty.*

Proof. For each n , choose an element $z_n \in F_n$. Then,

$$\begin{aligned} \forall \varepsilon > 0, \quad \exists N \text{ such that } \text{diam}(F_n) < \varepsilon \quad \forall n > N \\ \implies |z_n - z_m| < \varepsilon \quad \forall n, m > N \\ \implies (z_n)_n^\infty \text{ satisfies the Cauchy criterion} \\ \implies (z_n)_n^\infty \text{ converges to the same } z_0 \in \mathbb{C} \end{aligned}$$

for each $n, z_0 \in F_n$ since F_n is closed

$$\implies z_0 \in \bigcap_{n=1}^{\infty} F_n$$

□

Cauchy's Theorem (Triangular Path) 1. Let $f : A \rightarrow \mathbb{C}$ open, and f analytic. Suppose T is a triangular path in A with interior contained in A . Then:

$$\int_T f(z)dz = 0$$

Proof. Bisect the sides of T to form four smaller triangles $T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}$. Then:

$$\int_T f(z)dz = \sum_{j=1}^4 \int_{T^{(j)}} f(z)dz$$

(after cancellations)

$$\implies \left| \int_T f(z)dz \right| \leq \sum_{j=1}^4 \left| \int_{T^{(j)}} f(z)dz \right| \leq 4 \underbrace{\left| \int_{T^{(j)}} f(z)dz \right|}_{\text{(for some } j \text{)}}$$

Label this $T^{(j)}$ as T_1 . Repeat this procedure for T_1 : Construct four smaller triangles $T_1^{(1)}, T_1^{(2)}, T_1^{(3)}, T_1^{(4)}$. Again,

$$\int_{T_1} f(z)dz = \sum_{j=1}^4 \int_{T_1^{(j)}} f(z)dz \implies \left| \int_{T_1} f(z)dz \right| \leq 4 \underbrace{\left| \int_{T_1^{(j)}} f(z)dz \right|}_{\text{for some } j}$$

Label this $T_1^{(j)}$ as T_2 etc. for some j . Repeating this procedure we obtain a decreasing sequence of triangles $T_1 \supset T_2 \supset T_3 \supset \dots$

Let Δ_j denote T_j together with the interior of T_j , let Δ alone denote T with its interior. Then Δ_j is a decreasing sequence of closed sets.

$$\text{diam}\Delta_j = \frac{1}{2} \text{diam}\Delta_{j-1} = \frac{1}{2^j} \text{diam}\Delta \implies \lim_{j \rightarrow \infty} = 0$$

by the lemma,

$$\exists z_0 \in \bigcap_{j=1}^{\infty} \Delta_j$$

Since f is analytic at z_0 , given any $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\begin{aligned} 0 < |z - z_0| < \delta &\implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \\ &\implies |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0| \end{aligned}$$

Note that $f(z_0) - f'(z_0)(z - z_0)$ is a polynomial function and hence has a primitive.

$$\therefore \int_{T_j} f(z)dz = \int_{T_j} f(z) - f(z_0) - f'(z_0)(z - z_0) dz \quad \forall j$$

Choose T_j such that $\text{diam}T_j < \delta$. Then $|z - z_0| < \delta \quad \forall z \in T_j$

$$\begin{aligned} \implies \int_{T_j} f(z) dz &\leq \text{Length}(T_j) \cdot \sup_{z \in T_j} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \\ &< \text{Length}(T_j) \cdot \sup_{z \in T_j} \varepsilon |z - z_0| \\ &\leq \text{Length}(T_j) \cdot \text{diam}(T_j) \varepsilon \\ &= \frac{1}{2^j} \text{Length}(T) \cdot \frac{1}{2^j} \text{diam}(T) \varepsilon \end{aligned}$$

also note that:

$$\left| \int_T f(z) dz \right| \leq 4 \left| \int_{T_1} f(z) dz \right| \leq 4^j \left| \int_{T_j} f(z) dz \right|$$

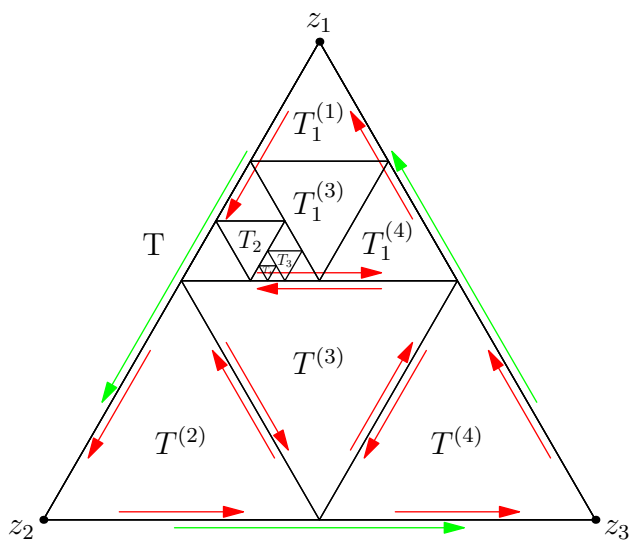
combining these two inequalities:

$$\left| \int_T f(z) dz \right| \leq \text{Length}(T) \cdot \text{diam}(T) \cdot \varepsilon$$

since ε is arbitrary

$$\implies \int_T f(z) dz = 0$$

□



Cauchy's Theorem (Star-shaped set) 1. Let $f : A \rightarrow \mathbb{C}$ be analytic. Let $A \subseteq \mathbb{C}$ be open and star-shaped. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a piecewise C^1 closed curve with $\gamma \in A$, then

$$\int_{\gamma} f(z)dz = 0$$

Proof. Suppose A is star-shaped about $w_0 \in A$. Define

$$F : A \rightarrow \mathbb{C} \quad z \mapsto \int_{[w_0; z]} f(\xi)d\xi$$

Claim: F is a primitive for f . Let $z_0 \in A$. Since A is open, $\exists D(z_0, r) \subset A$. If $z \in D(z_0, r)$, then the points z, z_0 and w_0 determine a triangular path T in A . We have:

$$\begin{aligned} F(z) - F(z_0) &= \left(\int_{[w_0; z]} f(\xi)d\xi - \int_{[w_0; z_0]} f(\xi)d\xi \right) = \left(\int_{[w_0; z]} f(\xi)d\xi + \int_{[z_0; w_0]} f(\xi)d\xi \right) \\ &= \left(\int_T f(\xi)d\xi - \int_{[z; z_0]} f(\xi)d\xi \right) = \int_{[z_0; z]} f(\xi)d\xi \quad (\text{by Cauchy thm. for triangle}) \end{aligned}$$

Note that $\int_{[z_0; z]} f(z_0)d\xi = f(z_0)(z - z_0)$, hence:

$$\begin{aligned} |F(z) - F(z_0) - f(z_0)(z - z_0)| &= \left| \int_{[z_0; z]} f(\xi) - f(z_0)d\xi \right| \\ &\leq \text{Length}([z_0; z]) \cdot \sup_{\xi \in [z_0; z]} |f(\xi) - f(z_0)| = |z - z_0| \cdot \sup_{\zeta \in [z_0; z]} |f(\zeta) - f(z_0)| \end{aligned}$$

f continuous at z_0 , $\therefore \exists \delta > 0$ such that $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$

$\therefore 0 < |z - z_0| < \min\{r, \delta\} \implies$

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \sup_{\zeta \in [z_0; z]} |f(\zeta) - f(z_0)| \leq \varepsilon$$

$\therefore F'(z_0) = f(z_0)$

This shows that F is a primitive for f

$$\implies \int_{\gamma} f(\zeta)d\zeta = \int_{\gamma} F'(\zeta)d\zeta = F(\gamma(b)) - F(\gamma(a)) = 0$$

□