MA1E01: Solutions week 9

Solution 1 *In order to find the vertical asymptotes we find the zeros of the denominator*

$$x^2 - 3x + 2 = 0 \Longrightarrow x = \frac{3 \pm \sqrt{9-8}}{2}$$

has two solutions, that are the candidates for vertical asymptotes: $x = 1, 2$. *In order to see it they really are vertical asymptotes, we evaluate the numerator. It is different from zero both at* $x = 1$ *and at* $x = 2$, *so we have two vertical asymptote.*

Moreover, since approaching 1 from the left the function $f(x)$ *increases, and approaching 1 from the right the function* $f(x)$ *decreases, we have*

$$\lim_{x \to 1^-} f(x) = +\infty$$
$$\lim_{x \to 1^+} f(x) = -\infty$$

Similarly

$$\lim_{x \to 2^-} f(x) = -\infty$$
$$\lim_{x \to 2^+} f(x) = +\infty$$

In order to check if $f(x)$ *has any horizontal asymptote, we compute dividing by* x^2

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1\frac{6x^3+3x+1}{x^2-3x+2} = \lim_{x\to\infty} \frac{6x+3/x+1/x^2}{1-3/x+2/x^2} = \lim_{x\to\infty} 6x+3/x+1/x^2$$

and this last limit does not exist. Similarly

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 1\frac{6x^3+3x+1}{x^2-3x+2} = \lim_{x\to-\infty} \frac{6x+3/x+1/x^2}{1-3/x+2/x^2} = \lim_{x\to-\infty} 6x+3/x+1/x^2$$

does not exist. So the function has no horizontal asymptote.

Solution 2 *It is obvious that* $\sqrt{2}$ *is the root of the function*

$$f(x) = x^2 - 2\,.$$

In fact the root is between 0 and 2, because $f(0) < 0 < f(2)$. *We are going to use Newton's method to find the root of* $f(x)$. *This reads*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}\,.$$

In our case

$$f'(x) = 2x\,.$$

We are going to start with $x_0 = 1$ *as an initial guess of the root. A few iterations of the* *method give*

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^2-2}{2x_0} = 1.000000 - \frac{(1.000000)^2-2}{2\times 1.000000} = 1.500000\\
x_2 &= x_1 - \frac{x_1^2-2}{2x_1} = 1.500000 - \frac{(1.500000)^2-2}{2\times 1.500000} = 1.466667\\
x_3 &= x_2 - \frac{x_2^2-2}{2x_2} = 1.466667 - \frac{(1.466667)^2-2}{2\times 1.466667} = 1.414216\\
x_4 &= x_3 - \frac{x_3^2-2}{2x_3} = 1.414216 - \frac{(1.414216)^2-2}{2\times 1.414216} = 1.414214\\
x_5 &= x_4 - \frac{x_4^2-2}{2x_4} = 1.414214 - \frac{(1.414214)^2-2}{2\times 1.414214} = 1.414214
\end{aligned}$$

And after 5 steps the method has converged to an approximation of $\sqrt{2}$ *that is good to 6 decimal places*

$$\sqrt{2} \approx 1.414214$$

Solution 3 *It is obvious that* $\sqrt[3]{7}$ *is the root of the function*

$$f(x) = x^3 - 7\,.$$

In fact the root is between 0 and 2, because $f(0) < 0 < f(2)$*. We are going to use Newton's method to find the root of* $f(x)$*. This reads*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}\,.$$

In our case

$$f'(x) = 3x^2\,.$$

We are going to start with $x_0 = 1$ *as an initial guess of the root. A few iterations of the method give*

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^3-7}{3x_0^2} = 1.000000 - \frac{(1.000000)^3-7}{3\times(1.000000)^2} = 3.000000\\
x_2 &= x_1 - \frac{x_1^3-7}{3x_1^2} = 3.000000 - \frac{(3.000000)^3-7}{3\times(3.000000)^2} = 2.259259\\
x_3 &= x_2 - \frac{x_2^3-7}{3x_2^2} = 2.259259 - \frac{(2.259259)^3-7}{3\times(2.259259)^2} = 1.963308\\
x_4 &= x_3 - \frac{x_3^3-7}{3x_3^2} = 1.963308 - \frac{(1.963308)^3-7}{3\times(1.963308)^2} = 1.914213\\
x_5 &= x_4 - \frac{x_4^3-7}{3x_4^2} = 1.914213 - \frac{(1.914213)^3-7}{3\times(1.914213)^2} = 1.912932\\
x_6 &= x_5 - \frac{x_5^3-7}{3x_5^2} = 1.912932 - \frac{(1.912932)^3-7}{3\times(1.912932)^2} = 1.912931\\
x_7 &= x_6 - \frac{x_6^3-7}{3x_6^2} = 1.912931 - \frac{(1.912931)^3-7}{3\times(1.912931)^2} = 1.912931
\end{aligned}$$

And after 7 steps the method has converged to an approximation of $\sqrt[3]{7}$ *that is good to 6 decimal places*

$$\sqrt[3]{7} \approx 1.912931$$

Solution 4 *First we determine the critical points by computing the derivative*

$$f'(x) = 6x^5 - 10x^4 + 4x^3 = x^3(6x^2 - 10x + 4)\,.$$

This derivative exists everywhere, so the only possible critical points are due to a vanishing derivative. To find them we solve

$$f'(x) = 0 \Longrightarrow \begin{cases} x^3 = 0 \\ or \\ 6x^2 - 10x + 4 = 0 \end{cases}$$

so either $x = 0$ *or*

$$x = \frac{10 \pm \sqrt{4}}{12} \Longrightarrow \begin{cases} x = 1 \\ x = 2/3 \end{cases}$$

We have 3 critical points $x = 0, 2/3, 1$. *They divide the real line in four regions, and we determine the sign of the derivative in each of them*

Region:	$x \in (-\infty, 0)$	$x \in (0, 2/3)$	$x \in (2/3, 1)$	$x \in (1, \infty)$
Sign of derivative:	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$

Therefore we get

$$f(0) = 0 \text{ is a minimum}$$
$$f(2/3) = 16/729 \text{ is a maximum}$$
$$f(1) = 0 \text{ is a minimum}$$

Finally we can determine the second derivative of the function

$$f''(x) = 30x^4 - 40x^3 + 12x^2$$

and we have

$$f''(0) = 0$$
$$f''(2/3) = -0.59... < 0$$
$$f''(1) = 2 > 0$$

And therefore the second derivative test confirms that $f(2/3)$ *is a maximum and* $f(1)$ *is a minimum, but is inconclusive for the critical point at* $x = 0$.

Solution 5 *First we determine the critical points by computing the derivative*

$$f'(x) = 5x^4 - 20x^2 = x^2(5x^2 - 20)\,.$$

This derivative exists everywhere, so the only possible critical points are due to a vanishing derivative. To find them we solve

$$f'(x) = 0 \Longrightarrow \begin{cases} x^2 = 0 \\ or \\ 5x^2 - 20 = 0 \end{cases}$$

so either $x = 0$ *or*

$$x = \pm 2$$

We have 3 critical points $x = -2, 0, 2$. *They divide the real line in four regions, and we* *determine the sign of the derivative in each of them*

Region:	$x \in (-\infty, -2)$	$x \in (-2, 0)$	$x \in (0, 2)$	$x \in (2, \infty)$
Sign of derivative:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$

Therefore we get

$$\begin{aligned} f(-2) &= 21.3... \text{ is a maximum} \\ f(0) &= 0 \text{ is neither a maximum nor a minimum} \\ f(2) &= -21.3... \text{ is a minimum} \end{aligned}$$

Finally we can determine the second derivative of the function

$$f''(x) = 20x^3 - 40x$$

and we have

$$\begin{aligned} f''(-2) &= 640 > 0 \\ f''(2/3) &= 0 \\ f''(2) &= -640 < 0 \end{aligned}$$

And therefore the second derivative test confirms that $f(-2)$ is a minimum and $f(2)$ is a maximum, but is inconclusive for the critical point at $x = 0$.