MA1E01: Solutions week 8

Solution 1

First we need to remember that the time needed to travel some distance d at speed v is d/v. In the graph we call x the horizontal distance travelled when we arrive at the interface between two mediums.

The total time of the travel is

$$T(x) = \frac{1}{v_1}\sqrt{h^2 + x^2} + \frac{1}{v_2}\sqrt{h^2 + (d-x)^2}\,.$$

In order to find the minimum time, we take the derivative and make it equal to zero

$$\frac{dT}{dx} = \frac{1}{v_1\sqrt{(h/x)^2 + 1}} - \frac{1}{v_2\sqrt{(\frac{h}{d-x})^2 + 1}} = 0\,.$$

In order to solve this equation, we need to work a bit the trigonometry. First we note the relations

$$\tan\theta_1 = \frac{x}{h}$$
$$\tan\theta_2 = \frac{d-x}{h}$$

So that

$$\frac{dT}{dx} = \frac{1}{v_1\sqrt{\frac{1}{\tan^2\theta_1} + 1}} - \frac{1}{v_2\sqrt{\frac{1}{\tan^2\theta_2} + 1}}\,.$$

A bit more of trigonometry shows that

$$\frac{1}{\tan^2\alpha} + 1 = \frac{\cos^2\alpha}{\sin^2\alpha} + 1 = \frac{\cos^2\alpha}{\sin^2\alpha} + \frac{\sin^2\alpha}{\sin^2\alpha} = \frac{\cos^2\alpha + \sin^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha}$$

therefore

$$\frac{dT}{dx} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}\,.$$

Finaly, equating this to zero, one obtains Snell law.