MA1E01: Solutions week 4

Solution 1

1. *Since we only consider values larger than 0 we do not have to worry about the* $\sqrt{x}$ *and we have*

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x}\,\frac{\sin x}{x} = 0 \times 1 = 0$$

2. *Since the function is not defined for* $x < 0$, *the limit does not exists.*

3. *Since one of the side limits does not exist, the total limit also does not exist.*

Solution 2

1. *Dividing by* $1/x^2$

$$\lim_{x\to\infty} \frac{3x+1}{2x^2-2x-4} = \lim_{x\to\infty} \frac{3/x+1/x^2}{2-2/x-4/x^2} = 0$$

2. *Dividing by* x^4

$$\lim_{x\to\infty} \frac{3+3/x^2+(4\sin x)/x^4+2/x^4}{2+3/x^2-2/x^4} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}$$

(Note that $\lim_{x\to\infty} \frac{\sin x}{x^4} = 0$ *because* $\sin x$ *is always between* -1 *and* 1, *while the denominator increases without bound. This can be made rigorous by using the sandwich theorem and the fact that*

$$\frac{-1}{x^4} \le \frac{\sin x}{x^4} \le \frac{1}{x^4}$$

and

$$\lim_{x\to\infty} \frac{-1}{x^4} = \lim_{x\to\infty} \frac{1}{x^4} = 0\,.$$

)

3. *Dividing by* x^2

$$\lim_{x\to\infty} \frac{3+3/x+(4\sin(x^7))/x^2-3/x^2}{3-2/x^2} = \lim_{x\to\infty} \frac{3}{3} = 1$$

(Note that $\lim_{x\to\infty} \frac{\sin(x^7)}{x^2} = 0$ *because* $\sin(x^7)$ *is always between* -1 *and* 1 *(the power of* x *does not change anything), while the denominator icreases without bound.)*

Solution 3 *The function is continuous if*

$$\lim_{x\to 0} f(x) = f(0) = 0\,.$$

In order to determine the limit, we use the sandwich theorem to compute each of the one-side limits. For $x > 0$ we have that

$$-x \le x\sin(1/x) \le x$$

And since both $\lim_{x\to 0^+} -x = \lim_{x\to 0^+} x = 0$, we have

$$\lim_{x\to 0^+} x\sin(1/x) = 0$$

For $x < 0$ the proof is similar, since now

$$x \le x\sin(1/x) \le -x$$

and $\lim_{x\to 0^-} -x = \lim_{x\to 0^-} x = 0$. Therefore

$$\lim_{x\to 0^-} x\sin(1/x) = 0\,.$$

By definition, since both one-side limits agree, the limit exist

$$\lim_{x\to 0} x\sin(1/x) = 0$$

and agrees with $f(0)$, so the function is continous at $x = 0$. At any other point the function is continous because it is the product of continuous functions (no division by zero is present for $x \neq 0$).

Solution 4 *The easiest way to determine the limit is to divide the recursive relation by a_n to obtain*

$$1 = \frac{a_{n-1}}{a_n} + \frac{a_{n-2}}{a_n} = \frac{a_{n-1}}{a_n} + \frac{a_{n-2}}{a_{n-1}}\frac{a_{n-1}}{a_n}\,.$$

Obviously

$$\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = \lim_{n\to\infty} \frac{a_{n-2}}{a_{n-1}}$$

Calling $x = \lim_{n\to\infty} \frac{a_{n-1}}{a_n}$, we have

$$1 = x + x^2$$

That is a second degree equation with solution

$$x = \lim_{n\to\infty} \frac{a_{n-1}}{a_n} = \frac{-1+\sqrt{5}}{2}\,. \tag{1}$$

The result does not depend on the initial conditions a_1 and a_2 (we have not used them to determine the result).