MA1E01: Solutions week 4

Solution 1 *We have to show that for every* $\epsilon > 0$, *there exist a* $\delta > 0$ *such that*

$$0 < |x-a| < \delta \Longrightarrow |f(x)g(x) - L_1L_2| < \epsilon\,. \tag{1}$$

First note that by hypothesis there exists numbers $\delta_1, \delta_2 > 0$ *such that*

$$0 < |x-a| < \delta_1 \Longrightarrow |f(x) - L_1| < \sqrt{\epsilon}\,.$$

and

$$0 < |x-a| < \delta_1 \Longrightarrow |f(x) - L_1| < \sqrt{\epsilon}\,.$$

In particular, choosing $\delta = \min(\delta_1, \delta_2)$ *we have*

$$0 < |x-a| < \delta \Longrightarrow |f(x) - L_1||g(x) - L_2| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon\,.$$

so we have proved that

$$\lim_{x\to a}[(f(x) - L_1)(g(x) - L_2)] = 0\,.$$

But now, with a bit of algebra

$$\begin{aligned}\lim_{x\to a}[(f(x) - L_1)(g(x) - L_2)] &= \lim_{x\to a}[f(x)g(x) - L_1g(x) - L_2f(x) + L_1L_2] = \\ &\lim_{x\to a}[f(x)g(x)] - L_1\lim_{x\to a}g(x) - L_2\lim_{x\to a}f(x) + \lim_{x\to a}L_1L_2 = \\ &\lim_{x\to a}[f(x)g(x)] - L_1L_2 - L_2L_1 + L_1L_2 = 0\,.\end{aligned}$$

As we wanted to show.

Solution 2

1. *Since* $\lim_{x\to 1} x+1 = 2 \neq 0$, *we have*

$$\lim_{x\to 1}\frac{3x-3}{x+1} = \frac{\lim_{x\to 1} 3x-3}{\lim_{x\to 1} x+1} = 0$$

2. *Since* $\lim_{x\to 1} x - 1 = 0$ *we have to be careful. The limit of the numerator is* $\lim_{x\to 1} 3x - 2 = 1 \neq 0$, *therefore the limit* $\lim_{x\to 1}\frac{3x-2}{x-1}$ *does not exist.*

3. *Since* $\lim_{x\to 1} x - 1 = 0$ *we have to be careful. The numerator has as limit* $\lim_{x\to 1}\sqrt{x+3} - 2 = 0$, *so we need to work. For* $x \neq 1$ *we have*

$$\frac{\sqrt{x+3}-2}{x-1} = \frac{\sqrt{x+3}-2}{x-1} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \frac{x-1}{x-1} \times \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{x+3}+2}\,.$$

Therefore

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1}\frac{1}{\sqrt{x+3}+2} = 0.25\,.$$

Solution 3

1. *Since* $\lim_{x\to 1} x - 1 = 0$ *we have to be careful. The limit of the numerator is* $\lim_{x\to 1} 3x - 3 = 0$, *so we have to operate algebraically. For* $x \neq 1$ *we have*

$$\frac{3x-3}{x-1} = 3\,.$$

and therefore

$$\lim_{x\to 1} \frac{3x-3}{x-1} = \lim_{x\to 1} 3 = 3\,.$$

2. *Since* $\lim_{x\to 1} x - 1 = 0$ *we have to be careful. The numerator has as limit* $\lim_{x\to 1} \sqrt{x^2+3} - 2 = 0$, *so we need to work. For* $x \neq 1$ *we have*

$$\frac{\sqrt{x^2+3}-2}{x-1} = \frac{\sqrt{x^2+3}-2}{x-1} \times \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} = \frac{x^2-1}{x-1} \times \frac{1}{\sqrt{x+3}+2} = \frac{x+1}{\sqrt{x+3}+2}\,.$$

Therefore

$$\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} = \lim_{x\to 1} \frac{x+1}{\sqrt{x+3}+2} = 0.5\,.$$