MA1E01: Solutions week 10

Solution 1 *The derivative is given by*

$$f'(x) = 3 - \cos x\,.$$

Now, since $-1 \leq \cos x \leq 1$*, it follows that* $f'(x) > 0$*. In particular the derivative is not zero anywhere.*

By Rolle's theorem, if f' *is* ***never*** *zero, there cannot be two points* a *and* b *such that* $f(a) = f(b)$*. In particular* $f(x)$ *can vanish at most at one point.*

Solution 2 *First let's show that* $f(1) \leq 10$*. This is a direct application of the mean value theorem: If* $f(1) > 10$*, then there would exist a poinit* $c \in (0, 1)$ *such that*

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = f(1) > 10$$

contradicting the hypotesis.

Now if $f(x) > 10$ *for some* $x \in (0, 1)$*, we also have a contradiction, since there would exists a* $c \in (0, x)$ *such that*

$$f'(c) = \frac{f(x) - f(0)}{x - 0} > f(x) - f(0) > 10\,.$$

Again a contradiction with the hypotesis.

Solution 3

1. *In the difference*

$$\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 =$$
$$[2^3 + 3^3 + \cdots + (n-1)^3 + n^3 + (n+1)^3] - [1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3]$$

everything cancels except the terms $(n+1)^3 - 1$*.*

Now we have that

$$\sum_{k=1}^{n}\Big[3k^2 + 3k + 1\Big] = (n+1)^3 - 1$$

or

$$3\sum_{k=1}^{n} k^2 = n^3 + 3n^2 + 3n - 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

By using $\sum_{k=1}^{n} 1 = n$*, and Gauss trick*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

we find the desired expression

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} \tag{1}$$

2. *Since the function $f(x) = x^2$ is monotonous and increasing in $[0,1]$, the maximum in the interval $[t_{k-1}, t_k]$ is $f(t_k)$, and the minimum $f(t_{k-1})$. So we have*

$$M_k = \{Max.\ of\ f(x)\ in\ [t_{k-1}, t_k]\} = f(t_k) = f(k/n) = (k/n)^2\,,$$
$$m_k = \{Min.\ of\ f(x)\ in\ [t_{k-1}, t_k]\} = f(t_{k-1}) = f((k-1)/n) = (k-1)^2/n^2\,.$$

Therefore the upper and lower sums are

$$U(f, \mathcal{P}_n) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 (t_k - t_{k-1}) = \frac{1}{n^3}\sum_{k=1}^{n} k^2$$
$$L(f, \mathcal{P}_n) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 (t_k - t_{k-1}) = \frac{1}{n^3}\sum_{k=1}^{n} (k-1)^2$$

Using equation (1) we get

$$\begin{aligned} U(f, \mathcal{P}_n) &= \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \\ L(f, \mathcal{P}_n) &= \frac{2n^3 - 3n^2 + n}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \end{aligned} \tag{2}$$

3. *Now it is easy to compute*

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{1}{n}$$

and therefore

   (a) *Yes, we can take for example $\mathcal{Q} = \mathcal{P}_{11}$, since*

   $$U(f, \mathcal{P}_{11}) - L(f, \mathcal{P}_{11}) = \frac{1}{11} = 0.09 < 0.1$$

   (b) *Yes, we can take for example $\mathcal{Q} = \mathcal{P}_{101}$, since*

   $$U(f, \mathcal{P}_{101}) - L(f, \mathcal{P}_{101}) = \frac{1}{101} = 0.0099 < 0.01$$

   (c) *Yes, we can take for example $\mathcal{Q} = \mathcal{P}_{1001}$, since*

   $$U(f, \mathcal{P}_{1001}) - L(f, \mathcal{P}_{1001}) = \frac{1}{1001} = 0.000999 < 0.001$$

Of course we could have taken $\mathcal{Q} = \mathcal{P}_{1001}$ for the three cases... or any other $\mathcal{P}_n$ with $n > 1000$.

4. *By definition the value of the integral is sandwiched between the upper and lower sums of any partition $\mathcal{P}$*

$$U(f, \mathcal{P}) \geq \int_0^1 x^2\, dx \geq L(f, \mathcal{P})\,.$$

If we choose as partition one of our $\mathcal{P}_n$, and write the values for $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ we get

$$\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \geq \int_0^1 x^2\, dx \geq \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\,.$$

Obviously the only possibly number for $\int_0^1 x^2\, dx$ that will obey this relation for all n is 1/3, so

$$\int_0^1 x^2\, dx = \frac{1}{3}\,.$$

(One can also take the $\lim_{n\to\infty}$ of $L(f, \mathcal{P}_n)$ and $U(f, \mathcal{P}_n)$ and check that both are 1/3).