MA1E01: Tutorial week 10

REMEMBER TO HAND BEFORE THE TUTORIAL STARTS

- Mean value theorem.
- Areas and partitions

Problem 1 *Prove that the function*

$$f(x) = 3x - \sin x + 7$$

has at most one root.

Problem 2 *A function $f(x)$ is continous and differentiable in $[0,1]$. If $f'(x) \leq 10$ for all $x \in [0,1]$ and $f(0) = 0$, what is the maximum possible value of $f(x)$ for x in $[0,1]$.*

NOTE: *If the "proof" seems difficult, try to think in this: "A car starts a journey at a speed that never exceeds 120km/h. What is the maximum distance that can be travelled in 1h?".*

Problem 3 *Our aim is to compute the intgeral*

$$\int_0^1 f. \qquad f(x) = x^2$$

in a complicated way.

1. *Give a proof that*

$$\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 = (n+1)^3 - 1 \tag{1}$$

(***Hint:*** *just write down a few terms, and you will see that most of them cancel in the diference).*

On the other hand

$$\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}\left[(k+1)^3 - k^3\right] = \sum_{k=1}^{n}\left[3k^2 + 3k + 1\right] \tag{2}$$

By using equations (1) and (2), show that

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} \tag{3}$$

2. *The partition*

$$\mathcal{P}_n = \{t_0, t_1, \ldots, t_n\} \tag{4}$$

with

$$t_k = \frac{k}{n} \tag{5}$$

divides the interval $[0,1]$ in n equal intervals of the form

$$[t_{k-1}, t_k] = \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

and size $1/n$

Find an expression for the Upper and Lower sums

$$U(f, \mathcal{P}_n)$$
$$L(f, \mathcal{P}_n)$$

(You will need the sum of the previous section here)

3. *Compute*

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) \tag{6}$$

Can you find partitions $\mathcal{Q}$ such that

(a)
$$U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < 0.1\,. \tag{7}$$

(b)
$$U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < 0.01\,. \tag{8}$$

(c)
$$U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < 0.001\,. \tag{9}$$

4. *Using the results of the previous sections, show that the integral that we want to compute is actually 1/3.*