Recap: Limits

- We write
  \[
  \lim_{x \to a} f(x) = L
  \]
  if \( f \) approaches \( L \) close to \( a \).

- The limit of \( f \) at \( a \) has nothing to do with the value of the function at \( a \). The limit can exist even if the function is not defined. For example:
  \[
  \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2,
  \]
  but the function \( \frac{x^2 - 1}{x - 1} \) is not even defined at \( x = 1 \). (See example 2, section 1.1).

- Right and left limits mean (see example 2, section 2.1):
  \[
  \lim_{x \to a^+} f(x) = L \implies f \text{ approaches } L \text{ close to } a \text{ for values larger than } a.
  \]
  \[
  \lim_{x \to a^-} f(x) = L \implies f \text{ approaches } L \text{ close to } a \text{ for values smaller than } a.
  \]

- A limit exists if and only if the right and left limits exist and are the same.

  \[
  \lim_{x \to a} f(x) = L \iff \begin{cases} 
  \lim_{x \to a^+} f(x) = L \\
  \lim_{x \to a^-} f(x) = L 
  \end{cases}
  \]

- If a function is continuous at \( a \), then the limit at \( a \) is given by the value of the function (Examples 1 of section 1.5).

  \[
  f(x) \text{ continuous at } a \implies \lim_{x \to a} f(x) = f(a).
  \]

- Polynomials and the trigonometric functions \( \sin x, \cos x \) are continuous at every point. The function \( \sqrt{x} \) is continuous at \( x > 0 \). The sum, difference and composition of continuous functions is continuous. The ratio of continuous functions \( f(x)/g(x) \) is continuous if \( g(x) \neq 0 \). (Examples 3, 4 of section 1.5).

- Computing limits.
  
  1. If the function is continuous, just evaluate the function. Example:

  \[
  \lim_{x \to 2} \sqrt{x^2 + 12x + 4} = 6.
  \]

- If the function is not continuous, is probably of the form \( \lim_{x \to a} f(x)/g(x) \) with \( \lim_{x \to a} g(x) = 0 \). In this case (see Example 9 section 1.2)
  - If \( \lim_{x \to a} f(x) \neq 0 \) the limit does not exists.
  - If \( \lim_{x \to a} f(x) = 0 \) one needs to simplify the expression, assuming that \( x \neq a \).

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)