Recap: Limits

- We write
$$\lim_{x\to a} f(x) = L$$
if **f approaches** L **close to** a.

- The limit of f at a has **nothing** to do with the value of the function at a. The limit can exists even if the function is not defined. For example:
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2\,,$$
but the function $\frac{x^2-1}{x-1}$ is not even defined at $x = 1$. (See example 2, section 1.1).

- Right and left limits mean (see example 2, section 2.1):
$$\lim_{x\to a^+} f(x) = L \quad\Longrightarrow\quad \text{f approaches } L \text{ close to } a \text{ for values larger than } a.$$
$$\lim_{x\to a^-} f(x) = L \quad\Longrightarrow\quad \text{f approaches } L \text{ close to } a \text{ for values smaller than } a.$$

- A limit exists **if and only if** the right and left limits exists **and** are the same.
$$\lim_{x\to a} f(x) = L \Longleftrightarrow \begin{cases} \lim_{x\to a^+} f(x) = L \\ \lim_{x\to a^-} f(x) = L \end{cases}$$

- If a function is continous at a, then the limit at a is given by the value of the function (Examples 1 of section 1.5).
$$f(x) \text{ continous at } a \Longrightarrow \lim_{x\to a} f(x) = f(a)\,.$$

- Polynomials and the trigonometric functions $\sin x, \cos x$ are continous at every point. The function $\sqrt{x}$ is continous at $x > 0$. The sum, difference and composition of continous functions is continous. The ratio of continous functions $f(x)/g(x)$ is continous if $g(x) \neq 0$. (Examples 3, 4 of section 1.5).

- Computing limits.

  1. If the function is continous, just evaluate the function. Example:
  $$\lim_{x\to 2} \sqrt{x^3 + 12x + 4} = 6\,.$$

- If the function is **not** continous, is probably of the form $\lim_{x\to a} f(x)/g(x)$ with $\lim_{x\to a} g(x) = 0$. In this case (see Example 9 section 1.2)

  - If $\lim_{x\to a} f(x) \neq 0$ the limit does not exists.
  - If $\lim_{x\to a} f(x) = 0$ one needs to simplify the expression, assuming that $x \neq a$.

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$