MA1E01: Solutions model exam I

Solution 1

(a) *Since the denominator $x^2 - 3x + 2 = (x-2)(x-1)$ vanish at $x = 1, 2$, the domain of $f(x)$ are all real numbers except $x = 1, 2$.*

(b) *Horizontal asymptotes*

$$\lim_{x\to\infty} \frac{2x^2-2}{x^2-3x+2} = \lim_{x\to\infty} \frac{2-2/x^2}{1-3/x+2/x^2} = \frac{\lim_{x\to\infty}(2-2/x^2)}{\lim_{x\to\infty}(1-3/x+2/x^2)} = \frac{2}{1} = 2$$

(and similar for the limit $x \to -\infty$). We have an horizontal asymptote at $y = 2$.

Vertical asymptotes: We have two candidates $x = 1, 2$ (where the denominator vanish). We compute

$$\lim_{x\to 1} \frac{2x^2-2}{x^2-3x+2} = \lim_{x\to 1} \frac{2(x-1)(x+1)}{(x-2)(x-1)} = \lim_{x\to 1} \frac{2(x+1)}{(x-2)} = -4$$

and therefore $x = 1$ is not a vertical asymptote-

The other candidate

$$\lim_{x\to 2} \frac{2x^2-2}{x^2-3x+2} = \lim_{x\to 2} \frac{2(x-1)(x+1)}{(x-2)(x-1)} = \pm\infty$$

is a vertical asymptote (at $x = 2$).

(c) *After the simplification*

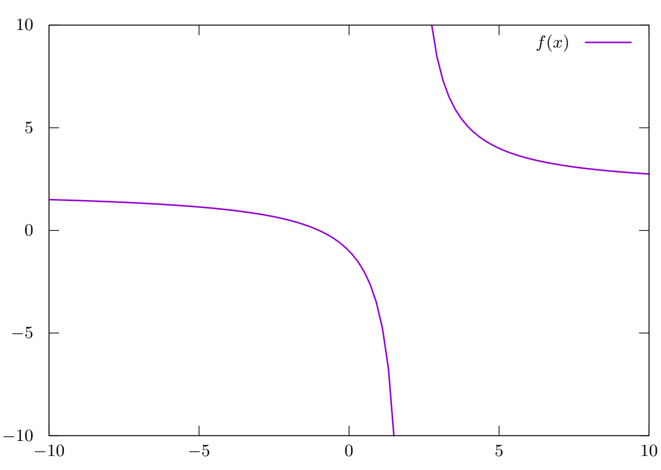
$$f(x) = \frac{2x^2-2}{x^2-3x+2} = \frac{2(x-1)(x+1)}{(x-2)(x-1)} = \frac{2(x+1)}{(x-2)}$$

(valid for $x \neq 1$) we can determine

$$f'(x) = \frac{2(x-2)-2(x+1)}{(x-2)^2} = \frac{-6}{(x-2)^2} \qquad (x \neq 1)\,.$$

This derivartive never vanishes, so the function has no maxima/minima.

(d) *The function looks like this:*



Solution 2 *It is convenient to put names to the different quantities*

- $S = 0.0625\,\mathrm{m}^2$ *is the total area of the paper.*
- $a = 0.05\,\mathrm{m}$ *is the right/left margin.*
- $b = 0.07\,\mathrm{m}$ *is the top/bottom margin.*
- h *is the total height of the paper (including top/bottom margins).*
- L *is the total width of the paper (including left/right margins).*

With these variables we have that the total surface of the paper is S, so that

$$Lh = S$$

and that we want to maximize the printing area of the page

$$\textit{printing area} = (L - 2a)(h - 2b) = \text{maximum}\,.$$

From the first equation, we get $L = S/h$. Substituting in the second equation

$$\left(\frac{S}{h} - 2a\right)(h - 2b) = S - \frac{2bS}{h} - 2ah + 4ab = \text{maximum}\,.$$

We have to determine the value of h that maximizes the printing area, so the condition is

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(S - \frac{2bS}{h} - 2ah + 4ab\right) = 0$$

or

$$\frac{2bS}{h^2} - 2a = 0 \Longrightarrow h = \sqrt{\frac{bS}{a}} = \sqrt{\frac{0.07 \times 0.0625}{0.05}}\,\mathrm{m}\,.$$

and substituting in $L = S/h$ we get

$$L = \sqrt{\frac{0.05 \times 0.0625}{0.07}}\,\mathrm{m}\,.$$

NOTE: *In the exam, without calculator, nobody expects you to do the square roots by hand. Leave results like this (which is perfectly correct).*

Solution 3 *Proof by contradiction. Imagine that $f(x)$ is such that*

$$\int_0^1 f(x)\,\mathrm{d}x > 1/2 \Longrightarrow \int_0^1 f(x) - x\,\mathrm{d}x > 0$$

Now define a new function

$$G(x) = \int_0^x f(t) - t\,\mathrm{d}t\,.$$

We know the following properties of this function

1. $G(0) = 0$ *and* $G(1) > 0$
2. $G'(x) = f(x) - x$ *(by the fundamental theorem of calculus),*

3. $G'(0) = f(0) = 0$ (assumption 1.-).

4. $G''(x) = f'(x) - 1 \leq 0$ for all $x \in [0,1]$. (assumption 2.-).

Now the reasonning goes

- *$G''(x) \leq 0$ is the same as saying that $G'(x)$ is decreasing for all $x \in [0,1]$. Since $G'(0) = 0$, this means that $G'(x) \leq 0$ for $x \in [0,1]$.*
- *But if $G'(x) \leq 0$ for all $x \in [0,1]$ then $G(x)$ is decreasing. Since $G(0) = 0$, this means that $G(x) \leq 0$ for $x \in [0,1]$.*
- *But this contradicts that $G(1) > 0$.*

Since by assuming that $G(1) > 0$ we have arrived to a contradiction, this assumption must be wrong, so $G(1) \leq 0$, but

$$G(1) = \int_0^1 f(t) - t \,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t - \int_0^1 t\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t - 1/2 \leq 0$$

And therefore the maximum possible value for the integral is in fact 1/2.

Solution 4 *The volume of the surface of revolution for the function $f(x)$ between $x = a$ and $x = b$ is given by*

$$V = \pi \int_a^b [f(x)]^2 \,\mathrm{d}x$$

In our case, we have

$$V = \pi \int_0^{\pi/2} \sin^2(x)\cos(x)\,\mathrm{d}x\,.$$

In order to solve this integral we first solve the indefinite integral

$$\int \sin^2(x)\cos(x)\,\mathrm{d}x\,.$$

Integrating by parts (one can also use the substitution $u = \sin x$), we have

$$\begin{aligned} f(x) &= \sin^2 x &\implies& f'(x) = 2\sin x\cos x \\ g'(x) &= \cos x &\implies& g(x) = \sin x \end{aligned}$$

We have

$$\int \sin^2(x)\cos(x)\,\mathrm{d}x = \sin^3 x - 2\int \sin^2(x)\cos(x)\,\mathrm{d}x\,.$$

We see that we get the same integral that we are trying to solve. Solving for the value of the integral

$$3\int \sin^2(x)\cos(x)\,\mathrm{d}x = \sin^3 x \Longrightarrow \int \sin^2(x)\cos(x)\,\mathrm{d}x = \frac{1}{3}\sin^3 x$$

Finally by the fundamental theorem of calculus

$$V = \pi \int_0^{\pi/2} \sin^2(x)\cos(x)\,\mathrm{d}x = \pi\left[\frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{\pi}{3}$$

NOTE: *In the exam, without calculator, nobody expects you to divide π by 3. Leave results like this (which is perfectly correct).*