SOLUTION TO EXAM MA1E01

This solution is provided for reference. I emphasize the key elements of each problem.

Problem 1 *Consider the function*

$$f(x) = \frac{x^2 - 1}{x^2 + 3x + 2}$$

5 marks *Determine the domain of the function and its roots.*

The denominator vanish at

$$x^2 + 3x + 2 = 0 \Longrightarrow x = -1, -2.$$

Therefore the domain is all real numbers except $x = -1, -2$.

The roots are determined by the values of x where the function vanish

$$f(x) = 0 \Longrightarrow x^2 - 1 = 0 \Longrightarrow x = \pm 1\,,$$

but $x = -1$ is not in the domain, so the function has only one root at $x = 1$.

KEY: *knowing what is the domain and the roots of a function.*

5 marks *Determine the asymptotes of the function.*

Vertical: Where the function approaches $\pm\infty$. Candidates are where the denominator vanish: $x = -1, -2$. We determine the limits by factorizing

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)(x+2)} = -2$$

so no vertical asymptote at $x = -1$ (the function approaches -2 close to $x = -1$, and not $\pm\infty$). On the other hand

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{(x-1)}{(x+2)} = \pm\infty$$

And we have a vertical asymptote at $x = -2$.

The horizontal asymptote (behavior of the function when $x \to \pm\infty$) is determined by the usual method of dividing by the higest power of numerator/denominator

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 3x + 2} = \lim_{x\to\infty} \frac{1 - 1/x^2}{1 + 3/x + 2/x^2} = 1$$

(and similar for $x \to -\infty$)

KEY: *Knowing that a function goes to $\pm\infty$ at a vertical asymptote, and that an horizontal asymptote is related with the behaviour at $x \to \pm\infty$.*

5 marks *Determine the local maxima and minima.*

We need to find the points at which the derivative vanish. It is easier to use the factorized form

$$f(x) = \frac{x-1}{x+2}$$

valid for $x \neq -1$. *The derivartive is given by the quotient rule*

$$f'(x) = \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

This derivative never vanish, so the function has no maxima/minima.

KEY: *Knowing that to determine the min/max one has to solve* $f' = 0$. *Evaluating the derivative at* $x = -1, -2$, *or doing anything that is not looking where the derivative vanish is a zero. Not using the quotient rule and making the derivatives of the denominator and numerator is also zero marks.*

5 marks *Draw a sketch of the function*

Problem 2 *Consider the function*

$$f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}.$$

10 marks *Determine*

$$\lim_{x \to 0} f(x).$$

The easiest answer is that the limit does not exist because the function is not defined for $x < 0$. *Therefore the left limit does not exists.*

But those that have determined the right limit using L'hopital (taking the derivative of numerator and denominator)

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin(\sqrt{x})}{\sqrt{x}} = \lim_{x \to 0^+} \frac{\cos(\sqrt{x})/(2\sqrt{x})}{1/(2\sqrt{x})} = \cos(0) = 1$$

or any other valid method, also get (almost) the full marks.

KEY: *Referring to L'hopital rule, and then using the quotient rule (instead of taking the derivative of numerator/denominator independently) is automatically zero marks. Saying that 0/0=0 or* $0/0$ *means the limit is undefined is automatically zero marks. Saying that* $\sin(0) = 1$ *is also automatically zero marks.*

10 marks *Determine*

$$\int f(x) \mathrm{d}x.$$

The integral is determined by the substitution

$$u = \sqrt{x} \Longrightarrow du = \frac{1}{2\sqrt{x}} dx$$

so that

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx = \int \sin u (2du) = -2\cos u = -2\cos\sqrt{x}$$

KEY: *Knowing substitutions. Making the integral of numerator/denominator separately is automatically zero marks. Canceling* $\sqrt{x}$ *is automatically zero marks. Using the product rule for derivatives to do the integral is automatically zero marks.*

10 marks *Determine*

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t\,.$$

By the fundamental theorem of calculus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t = f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}\,.$$

KEY: *Of course it is perfectly fine if you just make the derivative of the result of the previous question (but then the previous question must be correct).*

Problem 3 ***30 marks*** *We want to make a tin can of cylindrical shape (both ends included). If we have 400cm*^{2} *of aluminum, what are the radius and height of the cylinder so that the volume is maximized?*

The total surface of the tin can (both ends included) is

$$S = 2\pi r h + 2\pi r^2 = 400\,\mathrm{cm}^2$$

where r and h are the radius and height of the cylinder. The Volume of the cylinder is

$$V = \pi r^2 h\,.$$

This formula involves both r and h. In order to get the volume as a function only of r or h we use the first equation to get

$$h = \frac{S - 2\pi r^2}{2\pi r} \tag{1}$$

substituting into the equation of the volume we get the volume as a function of only one variable (r in this case)

$$V(r) = r\frac{S - 2\pi r^2}{2}$$

In order to maximize the volume we take the derivative and make it zero

$$V'(r) = \frac{S}{2} - 3\pi r^2 = 0 \Longrightarrow r = \sqrt{\frac{S}{6\pi}} = \sqrt{\frac{200}{3\pi}}\,\mathrm{cm}\,.$$

To get h we use eq. (1)

$$h = 2\sqrt{\frac{S}{6\pi}} = 2r = 2\sqrt{\frac{200}{3\pi}}\,\mathrm{cm}\,.$$

So the height of the cylinder has to be equal to the diameter in order to maximize the volume of the tin can.

KEY: *To maximize/minimize it is mandatory to get the volume as a function of only one variable* ***and*** *make the derivative zero. Any other manipulation with volumes/surfaces is wrong and gets zero marks.*

Problem 4 ***20 marks*** *Consider the function*

$$f(x) = (\sin x)(\cos x)$$

Determine the area between the function and the x-axes between $x = 0$ and $x = \pi/2$.

The areal below the curve is, by definition, the integral between 0 and $\pi/2$. So we have to determine

$$A = \int_0^{\pi/2} \sin x \cos x \, dx.$$

The integral is easily solved with the substitution[1]

$$u = \sin x \Longrightarrow du = \cos x \, dx \, .$$

We have

$$\int \sin x \cos x \, dx = \int u \, du = \frac{u^2}{2} = \frac{\sin^2 x}{2}.$$

So the area below the curve is

$$A = \int_0^{\pi/2} \sin x \cos x \, dx = \left[\frac{\sin^2 x}{2} \right]_0^{\pi/2} = \frac{\sin^2(\pi/2)}{2} - 0 = \frac{1}{2}$$

KEY: *Knowing that the area below a curve is the integral. If you do anything else, this is automatically zero marks. If you say that the integral of the product is the product of the integrals, this is also zero marks.*

[1] It can also be solved by integrating by parts, or by using the trigonometric relation $\sin x \cos x = \frac{1}{2} \sin(2x)$.