1 Derivatives (W05/1)

1.1 Remember Galileo

- Galileo realized that the distance travelled by a falling object is very well approximated by the function

$$d(t) = 5t^2 \tag{1}$$

- What is the average speed of an object around $t_0 = 1$? Distance traveled over time

$$v_{\rm av}(t_0, t_1) = \frac{d(t_0) - d(t_1)}{t_0 - t_1} = \frac{5t_0^2 - 5t_1^2}{t_0 - t_1} \tag{2}$$

- The speed **at** $t = 1$ was defined as

$$v(1) = \lim_{t_0 \to 1} v_{\rm av}(t_0, 1) = \lim_{t_0 \to 1} \frac{5t_0^2 - 5}{t_0 - 1} = \lim_{t_0 \to 1} \frac{5(t_0 - 1)(t_0 + 1)}{t_0 - 1} = \lim_{t_0 \to 1} 5(t_0 + 1) = 10 \tag{3}$$

- Speed at $t = 1$ is the slope of the tangent line at $t = 1$.
- We can compute the speed at any point t

$$v(t) = \lim_{t_0 \to t} v_{\rm av}(t_0, t) = \lim_{t_0 \to t} \frac{5t_0^2 - 5t^2}{t_0 - t} \tag{4}$$

- If we use $t_0 = t + h$, then $\lim_{t_0 \to t} = \lim_{h \to 0}$ and we have

$$v(t) = \lim_{h \to 0} v_{\rm av}(t + h, t) = \lim_{h \to 0} \frac{d(t + h) - d(t)}{t - h}\,. \tag{5}$$

- Problem: determine the tangent line to the curve $f(x) = 2/x$ at $x = 2$

$$m = \lim_{h \to 0} \frac{2/(2 + h) - 2/2}{h} = \lim_{h \to 0} \frac{2/(2 + h) - (2 + h)/(2 + h)}{h} = \lim_{h \to 0} \frac{-h/(2 + h)}{h} = -1/2 \tag{6}$$

So the line is of the form $y = -\frac{1}{2}x + n$ and it pass through the point $(2, 1)$, so

$$y = -\frac{1}{2}x + 2 \tag{7}$$

- Problem: determine the tangent line to the curve $f(x) = x^2$ at the points $x = 1, 2, 3, 4$.

2 Rules of derivation I (W05/2)

The basic definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\,, \tag{8}$$

is enough to derive the rules of differentiation:

$$[f \pm g]' = f' \pm g'\,, \tag{9}$$

$$[fg]' = f'g + fg'\,, \tag{10}$$

$$\left[\frac{f}{g}\right]' = \frac{f'g - fg'}{g^2}\,, \tag{11}$$

$$[kf]' = kf' \qquad (k = \text{constant})\,, \tag{12}$$

$$[f(g)]' = f'(g)g' \qquad (\text{chain rule})\,, \tag{13}$$

and the derivatives of elementary functions

$$\begin{aligned} [\sin x]' &= \cos x\,, & (14)\\ [\cos x]' &= -\sin x\,, & (15)\\ [x^n]' &= nx^{n-1} \qquad (n \neq 0)\,, & (16)\\ [k]' &= 0 \qquad (k = \text{constant})\,, & (17)\\ [\log x]' &= \frac{1}{x}\,. & (18) \end{aligned}$$

3 Rules of derivation II (W05/3)

The computation of a derivative always involve a limit of the type 0/0

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{\Delta x}\,. \tag{19}$$

Now calling $\Delta x = h$ and $\Delta f = f(x+h) - f(x)$ we can write

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{\mathrm{d}f}{\mathrm{d}x}\,. \tag{20}$$

In this alternative notation, the chain rule is almost self-evident

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}\,. \tag{21}$$