

1 Intuitive notion of limit (W03/1)

1.1 Speed of a falling object

- Galileo realized that the distance travelled by a falling object is very well approximated by the function

$$d(t) = 5t^2 \tag{1}$$

- What is the average speed of an object around $t = 1$? Distance traveled over time

$$v_{\text{av}}(t) = \frac{d(t) - d(1)}{t - 1} = \frac{5t^2 - 5}{t - 1} \tag{2}$$

$d(1)$	t	$d(t)$	$v_{\text{av}}(t)$
5	1.2	7.2	11.
5	1.1	6.05	10.5
5	1.01	5.1005	10.05
5	1.001	5.010005	10.005
5	0.999	4.990005	9.995
5	0.99	4.9005	9.95
5	0.9	4.05	9.5
5	0.8	3.2	9.
5	0.7	2.45	8.5

- One might be tempted to just cancel the zero in the numerator/denominator, but this cannot be done in general!

$$f(x) = \frac{5x^2 - 5}{(x - 1)^2} \tag{3}$$

x	$f(x)$
1.2	55.
1.1	105.
1.01	1005.
1.001	10005.
0.999	-9995.
0.99	-995.
0.9	-95.
0.8	-45.

- Similarities: Both functions $f(x)$ and $v_{\text{av}}(x)$ are not defined at $x = 1$ (i.e. $x = 1$ is not in the domain).
- Difference between two situations: v_{av} **approaches** 10 close to 1, but f does not approach any reasonable number close to 1.
- Definition of limit: We say that

$$\lim_{x \rightarrow a} f(x) = L \tag{4}$$

if f approaches L close to a .

- Computing the limit: Since we have

$$v_{av}(t) = \frac{5t^2 - 5}{t - 1} = \begin{cases} 5(t + 1) & t \neq 1 \\ \text{Not defined} & t = 1 \end{cases} \quad (5)$$

Obviously v_{av} approaches 10 close to 1!:

$$\lim_{t \rightarrow 1} \frac{5t^2 - 5}{t - 1} = \lim_{t \rightarrow 1} \frac{5(t + 1)(t - 1)}{t - 1} = \lim_{t \rightarrow 1} 5(t + 1) = 10. \quad (6)$$

On the other hand

$$f(x) = \frac{5x^2 - 5}{(x - 1)^2} = \frac{5(x + 1)}{(x - 1)}. \quad (7)$$

and close to $x = 1$ the function grows/decrease without bound (i.e. does not approach any number). We write

$$\lim_{x \rightarrow 1} f(x) = \text{Does not exist}. \quad (8)$$

2 Rigorous definition of limit (W03/2)

2.1 Definition

We say that

$$\lim_{x \rightarrow a} f(x) = L \quad (9)$$

if for all $\epsilon > 0$, there exist a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon. \quad (10)$$

2.2 Some examples

1. We say that $\lim_{x \rightarrow 1} 2x + 1 = 3$, because for every $\epsilon > 0$, choosing $\delta = \epsilon/2$, for $0 < |x - 1| < \delta = \epsilon/2$ we have

$$|f(x) - L| = |2x + 1 - 3| = 2|x - 1| < 2\epsilon/2 < \epsilon \quad (11)$$

1. We say that $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$, because for every $\epsilon > 0$, choosing $\delta = \epsilon$, for $0 < |x - 1| < \delta$ we have

$$|f(x) - L| = \left| \frac{x^2 - 1}{x - 1} - 2 \right| = |x - 1| < \epsilon \quad (12)$$

3 Rigorous definition of limit (W03/3)

3.1 A special limit

This limit has to be remembered

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1. \quad (13)$$

3.2 The sandwich theorem

If $f(x) \leq h(x) \leq g(x)$ and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = L$ then

$$\lim_{x \rightarrow a} h(x) = L. \quad (14)$$

3.3 Basic rules

These following rules apply to the computation of limits

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x). \quad (15)$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = [\lim_{x \rightarrow a} f(x)][\lim_{x \rightarrow a} g(x)]. \quad (16)$$

$$\lim_{x \rightarrow a} g(x) \neq 0 \implies \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}. \quad (17)$$

$$\lim_{x \rightarrow a} f^n(x) = [\lim_{x \rightarrow a} f(x)]^n. \quad (18)$$

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}. \quad (19)$$

In the last case we require that $\lim_{x \rightarrow a} f(x) > 0$ when n is even