A remark on Schwarz's topological field theory

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Abstract

The standard evaluation of the partition function Z of Schwarz's topological field theory results in the Ray–Singer analytic torsion. Here we present an alternative evaluation which results in $Z = 1$. Mathematically, this amounts to a novel perspective on analytic torsion: it can be formaly written as a ratio of volumes of spaces of differential forms which is formally equal to 1 by Hodge duality. An analogous result for Reidemeister combinatorial torsion is also obtained.

1 Introduction

Analytic torsion [1] arises in a quantum field theoretic context as (the square of) the partition function of Schwarz's topological field theory [2, 3, 4]. This has turned out to be an important result in topological quantum field theory; for example it is used to evaluate the semiclassical approximation for the Chern–Simons partition function [5, 6], which gives a QFT-predicted formula for an asymptotic limit of the Witten–Reshetikhin–Turaev 3-manifold invariant [7] since this invariant arises as the partition function of the Chern–Simons gauge theory on the 3-manifold [5]. (See also [8] for a review of Schwarz's topological field theory in a general context, and [9] for some explicit results in the case of hyperbolic 3-manifolds.)

The partition function, Z , of Scwarz's topological field theory is a priori a formal, mathematically ill-defined quantity and its evaluation [2, 3, 4] is by formal manipulations which in the end lead to a mathematically meaningful result: $Z = \tau^{1/2}$ where τ is the analytic torsion of the background manifold. In this paper we show (§2) that there is an alternative formal evaluation of the partition function which results in the trivial answer $Z = 1$. This result amounts to a novel perspective on analytic torsion: we find that it can be formally written as a certain ratio of volumes of spaces of differential forms which is formally equal to 1 by Hodge duality.

Reidemeister combinatorial torsion (R-torsion) [10, 1] arises as the partition function of a discrete version of Schwarz's topological field theory [11, 12]. This is of potential interest if one attempts to capture the invariants of topological QFT in a discrete, i.e. combinatorial, setting. In §3 an analogue of the above-mentioned result of §2 is derived for combinatorial torsion.

2 Schwarz's topological field theory and analytic torsion

We begin by recalling the evaluation of the partition function

$$Z = \frac{1}{V} \int \mathcal{D}\omega \, e^{-S(\omega)} \tag{2.1}$$

of Schwarz's topological field theory [2, 3, 4]. Here V is a normalisation factor to be specified below. The background manifold ("spacetime") M is closed, oriented, riemannian, and has odd dimension $n = 2m + 1$. For simplicity we assume m is odd; then the following variant of Schwarz's topological field theory can be considered [4]: The field $\omega \in \Omega^m(M, E)$ is an m-form on M with values in some flat $O(N)$ vectorbundle E over M. The action functional is

$$S(\omega) = \int_M \omega \wedge d_m \omega \,. \tag{2.2}$$

Here $d_p : \Omega^p \to \Omega^{p+1}$ ($\Omega^p \equiv \Omega^p(M, E)$) is the exterior derivative twisted by a flat connection on E (which we surpress in the notation) and a sum over vector indices is implied in (2.2) [1]. A choice of metric on M determines an inner product in each Ω^p , given in terms of the Hodge operator $*$ by

$$\langle \omega \, , \omega' \rangle = \int_M \omega \wedge * \omega' \tag{2.3}$$

[1]Note that (2.2) vanishes if m is even.

Using this the action (2.2) can be written as $S(\omega) = \langle \omega\,, *d_m\omega\rangle$. Let $\ker(S)$ denote the radical of the quadratic functional S and $\ker(d_p)$ the nullspace of d_p. Then $\ker(S) = \ker(d_m)$, and after decomposing the integration space in (2.1) as $\Omega^m = \ker(S) \oplus \ker(S)^{\perp}$ the partition function can be formally evaluated to get

$$Z = \frac{V(\ker(S))}{V}\,\det{}'((*d_m)^2)^{-1/4} = \frac{V(\ker(S))}{V}\,\det{}'(d_m^* d_m)^{-1/4} \tag{2.4}$$

(we are ignoring certain phase and scaling factors; see [13] for these). Here $V(\ker(S))$ denotes the formal volume of $\ker(S)$. The obvious normalisation choice, $V = V(\ker(S))$, does not preserve a certain symmetry property which the partition function has when S is non-degenerate [4]; therefore we do not use this but instead proceed, following Schwarz, by introducing a resolvent for S. For simplicity we assume that the cohomology of d vanishes, i.e. $\mathrm{Im}(d_p) = \ker(d_{p+1})$ for all p ($\mathrm{Im}(d_p)$ is the image of d_p). Then S has the resolvent

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \ldots \longrightarrow \Omega^{m-1} \xrightarrow{d_{m-1}} \ker(S) \longrightarrow 0 \tag{2.5}$$

which we use in the following to formally rewrite $V(\ker(S))$. The orthogonal decompositions

$$\Omega^p = \ker(d_p) \oplus \ker(d_p)^{\perp} \tag{2.6}$$

give the formal relations

$$V(\Omega^p) = V(\ker(d_p))\,V(\ker(d_p)^{\perp})\,. \tag{2.7}$$

The maps d_p restrict to isomorphisms $d_p : \ker(d_p)^{\perp} \xrightarrow{\cong} \ker(d_{p+1})$, giving the formal relations

$$V(\ker(d_{p+1})) = |\det{}'(d_p)|\,V(\ker(d_p)^{\perp})\,. \tag{2.8}$$

Combining (2.7)–(2.8) we get

$$V(\ker(d_{p+1})) = \det{}'(d_p^* d_p)^{1/2}\,V(\Omega^p)\,V(\ker(d_p))^{-1}\,. \tag{2.9}$$

Now a simple induction argument based on (2.9) and starting with $V(\ker(S)) = V(\ker(d_m))$ gives the formal relation

$$V(\ker(S)) = \prod_{p=0}^{m-1}\Big(\det{}'(d_p^* d_p)^{1/2}\,V(\Omega^p)\Big)^{(-1)^p}\,. \tag{2.10}$$

A natural choice of normalisation is now [2]

$$V = \prod_{p=0}^{m-1} V(\Omega^p)^{(-1)^p} . \tag{2.11}$$

Substituting (2.10)–(2.11) in (2.4) gives

$$Z = \left[\prod_{p=0}^{m-1} \det{}'(d_p^* d_p)^{\frac{1}{2}(-1)^p} \right] \det{}'(d_m^* d_m)^{-1/4} . \tag{2.12}$$

These determinants can be given well-defined meaning via zeta-regularisation [1], resulting in a mathematically meaningful expression for the partition function. As a simple consequence of Hodge duality we have $\det'(d_p^* d_p) = \det'(d_{n-p-1}^* d_{n-p-1})$ which allows to rewrite (2.12) as

$$Z = \tau(M\,, d)^{1/2} \tag{2.13}$$

where

$$\tau(M\,, d) = \prod_{p=0}^{n-1} \det{}'(d_p^* d_p)^{\frac{1}{2}(-1)^p} \tag{2.14}$$

is the Ray–Singer analytic torsion [1]; it is independent of the metric, depending only on M and d. This variant of Schwarz's result is taken from [4]; it has the advantage that the resolvent (2.5) is relatively simple. The cases where m need not be odd, and the cohomology of d need not vanish, are covered in [2, 3] (see also [4] for the latter case). Everything we do in the following has a straightforward extension to these more general settings, but for the sake of simplicity and brevity we have omitted this.

We now proceed to derive a different answer for Z to the one above. Our starting point is (2.13)–(2.14) which we consider as a formal expression for Z, i.e. we do not carry out the zeta regularisation of the determinants. Instead, we use (2.8) to formally write

$$\det{}'(d_p^* d_p)^{1/2} = \frac{V(\ker(d_{p+1}))}{V(\ker(d_p)^{\perp})} \tag{2.15}$$

[2]This choice can be motivated by the fact that, in an analogous finite-dimensional setting, the partition function then continues to exhibit a certain symmetry property which it has when S is non-degenerate [4].

Substituting this in (2.14) and using (2.7) we find [3]

$$\tau(M\,,d) = \frac{V(\Omega^1)\,V(\Omega^3)\dots V(\Omega^n)}{V(\Omega^0)\,V(\Omega^2)\dots V(\Omega^{n-1})} \tag{2.16}$$

Formally, the ratio of volumes on the r.h.s. equals 1 due to

$$V(\Omega^p) = V(\Omega^{n-p})\,. \tag{2.17}$$

This is a formal consequence of the Hodge star operator being an orthogonal isomorphism from Ω^p to Ω^{n-p}. (Recall $\langle *\omega\,, *\omega'\rangle = \langle\omega\,,\omega'\rangle$ for all $\omega\,,\omega' \in \Omega^p$.) This implies $Z = 1$ due to (2.13).

The formal relation (2.16) shows that analytic torsion can be considered as a "volume ratio anomaly": The ratio of the volumes on the r.h.s. of (2.16) is formally equal to 1, but when $\tau(M\,,d)$ is given well-defined meaning via zeta regularisation of (2.14) a non-trivial value results in general.

It is also interesting to consider the case where n is even: In this case, using (2.7)–(2.8) we get in place of (2.16) the formal relation

$$\frac{V(\Omega^0)\,V(\Omega^2)\dots V(\Omega^n)}{V(\Omega^1)\,V(\Omega^3)\dots V(\Omega^{n-1})} = \prod_{p=0}^{n-1} \det{}'(d_p^* d_p)^{\frac{1}{2}(-1)^p} = 1 \tag{2.18}$$

The last equality is an easy consequence of Hodge duality and continues to hold after the determinants are given well-defined meaning via zeta regularisation [1]. On the other hand, the ratio of volumes on the l.h.s. is no longer formally equal to 1 by Hodge duality.

3 The discrete analogue

Given a simplicial complex K triangulating M a discrete version of Scwarz's topological field theory can be constructed which captures the topological quantities of the continuum theory [11, 12]. The discrete theory uses $\widehat{K}$, the cell decomposition dual to K, as well as K itself. This necessitates a field doubling in the continuum theory

[3] This relation is obtained without any restriction on m, i.e. for arbitrary odd n.

prior to discretisation. An additional field ω' is introduced and the original action $S(\omega) = \langle \omega \,, *d_m\omega \rangle$ is replaced by the doubled action,

$$\widetilde{S}(\omega,\omega') = \Big\langle \begin{pmatrix} \omega \\ \omega' \end{pmatrix} , \begin{pmatrix} 0 & *d_m \\ *d_m & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix} \Big\rangle = 2\int_M \omega' \wedge d_m\omega \tag{3.1}$$

This theory (known as the abelian BF theory [8]) has the same topological content as the original one; in particular its partition function, $\widetilde{Z}$, can be evaluated in an analogous way to get $\widetilde{Z} = Z^2 = \tau(M,d)$. The discretisation prescription is [11, 12]

$$(\omega,\omega') \;\to\; (\alpha,\alpha') \in C^m(K) \times C^m(\widehat{K}) \tag{3.2}$$

$$\widetilde{S}(\omega,\omega') \;\to\; \widetilde{S}(\alpha,\alpha') = \Big\langle \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} , \begin{pmatrix} 0 & *^{\widehat{K}} d_m^{\widehat{K}} \\ *^K d_m^K & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \Big\rangle \tag{3.3}$$

Here $C^p(K) = C^p(K,E)$ is the space of p-cochains on K with values in the flat $O(N)$ vectorbundle E ; $d_p^K : C^p(K) \to C^{p+1}(K)$ is the coboundary operator twisted by a flat connection on E ; $C^q(\widehat{K})$ and $d_q^{\widehat{K}}$ are the corresponding objects for $\widehat{K}$; $*^K : C^p(K) \to C^{n-p}(\widehat{K})$ and $*^{\widehat{K}} : C^q(\widehat{K}) \to C^{n-q}(K)$ are the duality operators induced by the duality between p-cells of K and $(n-p)$-cells of $\widehat{K}$. The p-cells of K and $\widehat{K}$ determine canonical inner products in $C^p(K)$ and $C^p(\widehat{K})$ for each p, and with respect to these the duality operators are orthogonal maps. (The definitions and background can be found in [14]; see also [1] and [11].) As in §2 we are assuming that m is odd and that the cohomology of the flat connection on E vanishes: $H^*(M,E) = 0$. Then the partition function of the discrete theory, denoted $\widetilde{Z}_K$, can be evaluated by formal manipulations analogous to those in §2 (see [11, 12]) and the resulting expression can be written as either

$$\widetilde{Z}_K = \tau(K,d^K) \qquad \text{or} \qquad \widetilde{Z}_K = \tau(\widehat{K},d^{\widehat{K}}) \tag{3.4}$$

where

$$\tau(K,d^K) = \prod_{p=0}^{n-1} \det{}'(\partial_{p+1}^K d_p^K)^{\frac{1}{2}(-1)^p} \tag{3.5}$$

and $\tau(\widehat{K},d^{\widehat{K}})$ is defined analogously. Here ∂_{p+1}^K denotes the adjoint of d_p^K (it can be identified with the boundary operator on the $(p+1)$-chains of K). The quantities

$\tau(K, d^K)$ and $\tau(\widehat{K}, d^{\widehat{K}})$ coincide; in fact (3.5) is the Reidemeister combinatorial torsion (also called the R-torsion) of M determined by the given flat connection on E, and is the same for all cell decompositions K of M [10, 1]. (This is analogous to the metric-independence of analytic torsion.) Moreover, the analytic and combinatorial torsions coincide [15], so the discrete partition function in fact reproduces the continuum one:

$$\widetilde{Z}_K = \widetilde{Z} \tag{3.6}$$

We now present an analogue of the formal argument which led to $Z = 1$ in §2. Consider

$$\tau(K, d^K)\,\tau(\widehat{K}, d^{\widehat{K}}) = \prod_{p=0}^{n-1} \det{}'(\partial^K_{p+1} d^K_p)^{\frac{1}{2}(-1)^p} \det{}'(\partial^{\widehat{K}}_{p+1} d^{\widehat{K}}_p)^{\frac{1}{2}(-1)^p}. \tag{3.7}$$

Using the analogues of (2.15) and (2.7) in the present setting,

$$\det{}'(d_p^* d_p)^{1/2} = \frac{V(\ker(d^K_{p+1}))}{V(\ker(d^K_p)^\perp)} \tag{3.8}$$

and

$$V(C^p(K)) = V(\ker(d^K_p))\, V(\ker(d^K_p)^\perp), \tag{3.9}$$

and the corresponding $\widehat{K}$ relations, we find an analogue of the formal relation (2.16):

$$\begin{aligned} &\tau(K, d^K)\,\tau(\widehat{K}, d^{\widehat{K}}) \\ = \;& \frac{V(C^1(K))\,V(C^3(K))\ldots V(C^n(K))}{V(C^0(K))\,V(C^2(K))\ldots V(C^{n-1}(K))}\,\frac{V(C^1(\widehat{K}))\,V(C^3(\widehat{K}))\ldots V(C^n(\widehat{K}))}{V(C^0(\widehat{K}))\,V(C^2(\widehat{K}))\ldots V(C^{n-1}(\widehat{K}))} \end{aligned} \tag{3.10}$$

Formally, the r.h.s. equals 1 due to

$$V(C^p(K)) = V(C^{n-p}(\widehat{K})). \tag{3.11}$$

This is a formal consequence of the duality operator being an orthogonal isomorphism from $C^p(K)$ to $C^{n-p}(\widehat{K})$ (i.e. $\langle *^K\alpha, *\alpha'\rangle = \langle \alpha, \alpha'\rangle$ for all $\alpha, \alpha' \in C^p(K)$). This implies that, formally,

$$\widetilde{Z}_K = [\tau(K, d^K)\,\tau(\widehat{K}, d^{\widehat{K}})]^{1/2} = 1. \tag{3.12}$$

Thus we see that combinatorial torsion can also be considered as a "volume ratio anomaly" in an analogous way to analytic torsion.

Finally, in the n even case it is straightforward to find a combinatorial analogue of the formal relation (2.18) –we leave this to the reader.

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