Abelian Duality

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Abstract

We show that on three–dimensional Riemannian manifolds without boundaries and with trivial first real de Rham cohomology group (and in no other dimensions) scalar field theory and Maxwell theory are equivalent: the ratio of the partition functions is given by the Ray–Singer torsion of the manifold. At the level of interaction with external currents, the equivalence persists provided there is a fixed relation between the charges and the currents.

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There has been recent interest in relating different theories and establishing their equivalence. Common to all applications of the different aspects of the notion of duality is the observation that when two different theories are dual to each other, then either the manifolds are changed or the fields and the coupling constants are related.
In this paper we re-examine two simple systems — scalar field theory and Maxwell theory on three–dimensional Riemannian manifolds without boundaries and with trivial $\pi_1(\mathrm{M})$. We show the equivalence between these theories and we give the condition which must be satisfied by the charges and the external currents in order that the equivalence persists at the level of interactions. This is done by a direct calculation of the partition function of each theory paying particular attention to the fine structure of the zero–mode sector. In the spirit of Schwarz's method of invariant integration [1] we show that the ratio of the partition functions of the theories is equal to the square of the partition function of Chern–Simons theory (or the partition function of BF theory, that is, $\mathrm{U}(1)\times\mathrm{U}(1)$ Chern–Simons theory with purely off-diagonal coupling). Such equivalence between a scalar and vector theory is a novel form of duality which we call Abelian Duality. We show that when the coupling constants (overall scaling factors) are related as $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$, then this Abelian Duality transforms into $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$ duality. In this case the ratio of the partition functions is given by a topological invariant — the Ray–Singer torsion of the manifold. We show how our results can be obtained by Schwarz's resolvent method [2] and we use a resolvent generated by the de Rham complex to comment on possibilities of equivalence between the theories in other dimensions. In our considerations we use zeta–regularised determinants.

We will consider Riemannian manifolds M. All operators entering our theory can be described by the following diagram:

$$
\begin{array}{ccc}
\Omega^{p}(\mathrm{M}) & \xrightarrow{\;d_p\;} & \Omega^{p+1}(\mathrm{M}) \\
\uparrow * & & \downarrow * \\
\Omega^{m-p}(\mathrm{M}) & \xrightarrow{\;d^{\dagger}_{m-p-1}\;} & \Omega^{m-p-1}(\mathrm{M})
\end{array}
\qquad (1)
$$

where $m = \dim \mathrm{M}$. The case of interest will be a three-dimensional manifold.

We will first calculate the partition function of free Maxwell theory:

$$
\begin{aligned}
Z_1(\lambda_1) &= \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{-i\lambda_1 \int d^3x \sqrt{g}\; F_{\mu\nu}F^{\mu\nu}} = \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{-i\lambda_1 \int d_1A \wedge * d_1A} \\
&= \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{-i\lambda_1 \langle d_1A,\; d_1A\rangle} = \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{-i\lambda_1 \langle A,\; d_1^\dagger d_1A\rangle}. \qquad (2)
\end{aligned}
$$

Note that here we have written F as d_1A. The Maxwell equation $d_2F = 0$ implies that F is an element in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$. A is a one-form and therefore $d_1A = 0$ in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$, that is, if $F = d_1A$ then the equivalence class $[F]$ is zero in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$ (i.e. $[F] = [F'] \iff F = F' + d_1A$). When the second de Rham cohomology group is trivial, then $F = d_1A$ is valid globally. In three dimensions $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$ is isomorphic to $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M})$ due to Hodge duality and $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M})$ being trivial means that the first homotopy group $\pi_1(\mathrm{M})$ is trivial (then $F = d_1A$ globally). If $\pi_1(\mathrm{M})$ is non-trivial, then $F = d_1A$ is valid only on contractible regions of the manifold.
We will show the equivalence between Maxwell theory and scalar field theory for the case when F could be written as d_1A globally, that is, for homology 3-spheres $\big($i.e. manifolds with trivial first real de Rham cohomology group, e.g. $\mathbf{S}^3$ or the lens spaces $\mathrm{L}(p,q)\big)$. For the general case the duality is more subtle and we would like to refer the reader to [3] where Witten has shown how to pass from scalar field theory to Maxwell theory and vice versa in two and three dimensions.
We need manifolds without boundaries because we want to integrate by parts to bring the differential operator d_1 on the other side of the scalar product and at the same time not to be bothered about boundary terms.
We need Riemannian manifolds, because we are dealing with partition functions.
The integral is over the space of all 1-forms $\Omega^1(\mathrm{M})$. We can decompose the space of all 1-forms as a direct sum of the kernel of the operator entering the partition function and its orthogonal complement:

$$\Omega^1(\mathrm{M}) = \ker d_1 \oplus (\ker d_1)^\perp. \qquad (3)$$

Therefore

$$Z_1(\lambda_1) = \mathrm{vol}(\ker d_1) \int\limits_{(\ker d_1)^\perp} \mathcal{D}A \; e^{-i\lambda_1 \langle A,\; d_1^\dagger d_1 A\rangle} = \mathrm{vol}(\ker d_1)\; \det{}'\Big(\frac{i\lambda_1}{\pi}\, d_1^\dagger d_1\Big)^{-1/2}. \qquad (4)$$

The partition function is thus an ill-defined quantity — the determinant and the volume factor are infinite. Our calculations will be formal. With a zeta–regularization technique we can make the determinant finite. We can also

asssume that an appropriate normalization is chosen in such way that the divergency of the volume factor is absorbed. This will make the partition function finite.
Note that vol$(\ker d_1)$ is nothing else but the Faddeev–Popov ghost determinant times the ghost–for–ghost determinant. To see that, let us calculate the same partition function using the method of invariant integration [1]. In other words we will exploit the gauge symmetry of the theory to restrict the integration over $\Omega^1(\mathrm{M})$ to integration over a lower-dimensional space — the space of the orbits of the group of gauge transformation.
The stabilizer of the group of gauge transformations $A \longrightarrow A + d_0\Omega^0(\mathrm{M})$ consists of those elements of $\Omega^0(\mathrm{M})$ for which $d_0\Omega^0(\mathrm{M}) = 0$, that is, the constant functions. In order to pick one representative of each equivalence class $[A]$, we impose a gauge condition, that is, we intersect the space of the orbits of the group of gauge transformations in the space of all 1-forms by a hyperplane defined by those A's, for which $\partial_\mu A^\mu = 0$, i.e. $d_0^\dagger A = 0$. The integration is then performed over this hyperplane. We can always make sure that the element at each intersection point of a group orbit with this hyperplane is not a zero–mode of the operator. Thus, the Faddeev–Popov trick not only restricts the gauge freedom, but also isolates the zero–modes of the operator. The Faddeev–Popov determinant is a delta function of the gauge-fixing condition (Lorentz gauge in our case). In addition we must multiply by the volume of each orbit in order to preserve the value of the partition function. That is, we must divide by the volume of the stabilizer of the group at each point (we have assumed that the volume of the group of gauge transformations is normalized to one). This volume factor is the ghost–for–ghost determinant — as the Faddeev–Popov determinant is not finite [4] — d_0 itself has zero–modes (it vanishes on the constant functions). So we need an analogue of the gauge-fixing condition — this time for the ghosts, not for the fields.
Assuming that all stabilizers are conjugate, we get:

$$Z_1(\lambda_1) \;=\; \frac{1}{\mathrm{vol}\,(\ker d_0)} \int\limits_{\Omega^1(M)/d_0} \mathcal{D}[A]\; e^{-i\,\lambda_1\,\langle A,\; d_1^\dagger d_1 A\rangle}\; \det{}'\,(d_0^\dagger d_0)^{1/2}. \qquad (5)$$

The stabilizer of the group of gauge transformations consists of the constant functions, that is, the stabilizer is the real line. The real line can be canonically identified with the zeroth de Rham cohomology group $\mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})$. The projection map $\ker d_q \longrightarrow \mathrm{H}^q_{\mathrm{dR}}(\mathrm{M})$ induces the isomorphism [4]:

$$\phi_q : \mathcal{H}^q(\mathrm{M}) \longrightarrow \mathrm{H}^q_{\mathrm{dR}}(\mathrm{M}) \qquad (6)$$

where $\mathcal{H}^q(\mathrm{M})$ is the space of harmonic q-forms. Therefore:

$$\mathrm{vol}\left(\mathcal{H}^q(\mathrm{M})\right) = |\det \phi_q|^{-1} \ \mathrm{vol}\left(\mathrm{H}^q_{\mathrm{dR}}(\mathrm{M})\right). \tag{7}$$

So the volume of the stabilizer is:

$$\mathrm{vol}(\ker d_0) = \det(\phi_0^\dagger \phi_0)^{1/2} \ \mathrm{vol}\left(\mathcal{H}^0(\mathrm{M})\right). \tag{8}$$

The volume of the orbit of the group is proportional to the ghost–for–ghost determinant $\det(\phi_0^\dagger\phi_0)$ that extracts the zero modes from the Faddeev–Popov ghost determinant $\det(d_1^\dagger d_1)$. The ghost–for–ghost determinant is equal to the inverse of the volume of the manifold [4]:

$$\det(\phi_0^\dagger\phi_0)^{-1} = \mathrm{vol}(\mathrm{M}). \tag{9}$$

Now we will extract the complex scaling factor $\frac{i\lambda_1}{\pi}$ from the functional determinant. Following [5] we can write:

$$\det{}'\left(\frac{i\lambda_1}{\pi}\, d_1^\dagger d_1\right)^{-1/2} \ = \ e^{-\frac{i\pi}{4}\eta(0,\, d_1^\dagger d_1)} \left(\frac{\lambda_1}{\pi}\right)^{-\frac{1}{2}\zeta(0,\, d_1^\dagger d_1)} \det{}'(d_1^\dagger d_1)^{-1/2}. \tag{10}$$

Thus the partition function of Maxwell theory is given by:

$$Z_1(\lambda_1) \ = \ e^{-\frac{i\pi}{4}\eta(0,\, d_1^\dagger d_1)} \left(\frac{\lambda_1}{\pi}\right)^{-\frac{1}{2}\zeta(0,\, d_1^\dagger d_1)} \frac{\mathrm{vol}(\mathrm{M})^{1/2}}{\mathrm{vol}\left(\mathcal{H}^0(\mathrm{M})\right)} \frac{\det{}'(d_0^\dagger d_0)^{1/2}}{\det{}'(d_1^\dagger d_1)^{1/2}}. \tag{11}$$

We now use the fact that on odd-dimensional and two-dimensional manifolds there are no poles in the ζ–function near $s = 0$. This can be seen using Seeley's formula [6] for the ζ-function of some Laplace-type operator L on a d-dimensional manifold without a boundary:

$$\zeta(s, L) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{A_n}{s + n - \frac{d}{2}} + \frac{J(s)}{\Gamma(s)}, \tag{12}$$

where A_n are the heat-kernel co-efficients and $J(s)$ is analytic. Then $\zeta(0,\, \Delta_p) = -\dim \mathrm{H}^p_{\mathrm{dR}}(\mathrm{M})$. Using the formula [5]:

$$\zeta(s,\, d_p^\dagger d_p) \ = \ (-1)^p \ \sum_{q=0}^{p} (-1)^q \ \zeta(s,\, \Delta_q), \tag{13}$$

we finally get, modulo the phase factor:

$$Z_1(\lambda_1) \ = \ \lambda_1^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \ \frac{\mathrm{vol}(\mathrm{M})^{1/2}}{\mathrm{vol}\left(\mathcal{H}^0(\mathrm{M})\right)} \ \det{}'(d_0^\dagger d_0)^{1/2} \ \det{}'(d_1^\dagger d_1)^{-1/2}. \tag{14}$$

Consider now the partition function of free scalar theory:

$$\begin{aligned} Z_0(\lambda_0) &= \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \;\; e^{-i\,\lambda_0 \int d^3x\,\sqrt{g}\;\partial_\mu\varphi\,\partial^\mu\varphi} = \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \;\; e^{-i\,\lambda_0 \int d_0\varphi\wedge * d_0\varphi} \\ &= \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \;\; e^{-i\,\lambda_0\,\langle d_0\varphi,\, d_0\varphi\rangle} = \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \;\; e^{-i\,\lambda_0\,\langle\varphi,\, d_0^\dagger d_0\varphi\rangle}. \end{aligned} \tag{15}$$

We now decompose the space of all 0-forms $\Omega^0(M)$ in a similar way:

$$\Omega^0(M) = \ker d_0 \oplus (\ker d_0)^\perp. \tag{16}$$

With this decomposition the partition function becomes:

$$\begin{aligned} Z_0(\lambda_0) &= \mathrm{vol}(\ker d_0) \int\limits_{(\ker d_0)^\perp} \mathcal{D}\varphi \;\; e^{-i\,\lambda_0\,\langle\varphi,\, d_0^\dagger d_0\varphi\rangle} \\ &= e^{-\frac{i\pi}{4}\eta(0,\,d_0^\dagger d_0)} \left(\frac{\lambda_0}{\pi}\right)^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \mathrm{vol}(\ker d_0)\;\det{}'(d_0^\dagger d_0)^{-1/2} \\ &= e^{-\frac{i\pi}{4}\eta(0,\,d_0^\dagger d_0)} \left(\frac{\lambda_0}{\pi}\right)^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \frac{\mathrm{vol}\big(\mathcal{H}^0(\mathrm{M})\big)}{\mathrm{vol}(\mathrm{M})^{1/2}}\;\det{}'(d_0^\dagger d_0)^{-1/2}. \end{aligned} \tag{17}$$

Modulo the phase factor we have:

$$Z_0(\lambda_0) = \lambda_0^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \frac{\mathrm{vol}\big(\mathcal{H}^0(\mathrm{M})\big)}{\mathrm{vol}(\mathrm{M})^{1/2}}\;\det{}'(d_0^\dagger d_0)^{-1/2}. \tag{18}$$

The product of the partition functions of the theories is:

$$Z_0(\lambda_0)Z_1(\lambda_1) = \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \det{}'(d_1^\dagger d_1)^{-1/2}. \tag{19}$$

On the other hand we have:

$$Z_1(\lambda_1) = \lambda_1^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})}\;\mathrm{vol}(\ker d_1)\;\det{}'(d_1^\dagger d_1)^{-1/4}\;\det{}'(d_1^\dagger d_1)^{-1/4}. \tag{20}$$

The Hodge star operator is invertible and on three–dimensional manifolds we have: $\det{}'(d_1^\dagger d_1)^{1/2} = \det{}'(*d_1)$. (For these operators the multiplicative anomaly vanishes.) Thus (modulo a phase factor):

$$Z_1(\lambda_1) = \lambda_1^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})}\;\det{}'(d_1^\dagger d_1)^{-1/4}\;\mathrm{vol}\big(\ker(*d_1)\big)\;\det{}'(*d_1)^{-1/2}. \tag{21}$$

The last two factors in this formula are exactly the partition function of Chern–Simons theory Z_{CS}. The partition function of Chern–Simons theory is a topological invariant (modulo a phase factor [5]), given by the Ray–Singer torsion of the manifold [2]:

$$Z_{CS}(\lambda_{CS}) \ = \lambda_{CS}^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \ \tau_{\mathrm{RS}}^{1/2}\,(\mathrm{M}). \tag{22}$$

Therefore:

$$\left(\frac{Z_1(\lambda_1)}{Z_{CS}(\lambda_{CS})}\right)^2 \ = \ \lambda_1^{-\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \ \det{}'(d_1^\dagger d_1)^{-1/2}\,. \tag{23}$$

Dividing (23) by (19) we get*:

$$\frac{Z_1(\lambda_1)}{Z_0(\lambda_0)} \ = \ Z_{CS}^2\left(\sqrt{\lambda_0\lambda_1}\ \right) \ = \ (\lambda_0\lambda_1)^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \ \tau_{\mathrm{RS}}(\mathrm{M}). \tag{24}$$

R $\longleftrightarrow$ 1/R duality means that if the coupling constants (overall scaling factors) are related as $\lambda_0 = \lambda_1^{-1}$ then both partition functions will depend on the coupling constants in the same way (one has to be careful, because the coupling constants are not dimensionless). The ratio of the partition functions (modulo an omitted phase factor) is a topological invariant — the Ray–Singer torsion of the manifold. Therefore the two theories are equivalent. For manifolds for which the Ray–Singer torsion is one ($\mathbf{S}^3$ for instance), the partition functions are equal.

Note that both scalar field theory and Maxwell theory are non-topological in three dimensions.

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$$\frac{Z_1(\lambda_1)}{Z_0(\lambda_0)} \ = \ Z_{CS}^2\left(\sqrt{\lambda_0\lambda_1}\ \right) \tag{25}$$

is stronger than R $\longleftrightarrow$ 1/R duality

$$\frac{Z_1(\lambda_1)}{Z_0\left(\frac{1}{\lambda_1}\right)} \ = \ \tau_{\mathrm{RS}}(\mathrm{M}) \tag{26}$$

*After this work was completed, our attention was kindly drawn by A. Schwarz to [7] where the ratio Z_{k-1}/Z_{m-k-1} (where m is the dimension of the manifold) is expressed as the Ray–Singer torsion. The difference between our work and [7] is in the following. In [7], the initial considerations are for the case when there are no zero modes of the Laplace operators Δ_k (acting on k-forms). When these zero modes are absent, it is rather obvious that the quotient Z_{k-1}/Z_{m-k-1} is the Ray–Singer torsion of the manifold. The case of interest appears when these zero modes are no longer neglected. In [7] a very deep analysis is given for this case: the theory of the measure of the path integrals involved is developed and certain general results are given in this direction. In our paper we have kept these zero modes all along and we have shown that even with them the quotient (Z_1/Z_0 in our case) is still given by the Ray–Singer torsion. In addition we have studied the scaling dependence of the models and we have shown the relation to R $\longleftrightarrow$ 1/R duality. We have also given treatment on the physically relevant case — interaction with external currents and correlation functions.

in the sense that if the coupling constants are not related as $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$, there is still a relation — the ratio of the partition functions is given by the square of the partition function of Chern–Simons theory with coupling constant $\lambda_{CS} = \sqrt{\lambda_1 \lambda_0}$, that is, by the partition function of $\mathrm{U}(1) \times \mathrm{U}(1)$ Chern–Simons theory with purely off-diagonal coupling (BF theory).

We can show the Abelian Duality by considering the following resolvent generated by the de Rham complex:

$$0 \longrightarrow \mathrm{I\!R} \xrightarrow{\phi_0^{-1}} \Omega^0(\mathrm{M}) \xrightarrow{d_0} \Omega^1(\mathrm{M}) \xrightarrow{d_1} \ker d_1 = \ker S_1 \longrightarrow 0. \quad (27)$$

The first cohomology group of the mainfold is trivial and we have:

$$\mathrm{vol}\,(\ker d_1) = \mathrm{vol}\,(\mathrm{Im}\, d_0). \quad (28)$$

If we denote by $\tilde{d}_0$ the restriction of d_0 over $(\ker d_0)^{\perp}$, then the map

$$\tilde{d}_0 : (\ker d_0)^{\perp} \longrightarrow \mathrm{Im}\, d_0 \quad (29)$$

implies:

$$\mathrm{vol}\,(\mathrm{Im}\, d_0) = \det{}'\,(d_0^{\dagger} d_0)^{1/2}\, \mathrm{vol}\Big((\ker d_0)^{\perp}\Big) = \det{}'\,(d_0^{\dagger} d_0)^{1/2}\, \frac{\mathrm{vol}\Big(\Omega^0(\mathrm{M})\Big)}{\mathrm{vol}\,(\ker d_0)}. \quad (30)$$

We thus get:

$$\mathrm{vol}\,(\ker d_1) = \mathrm{vol}\Big(\Omega^0(\mathrm{M})\Big)\, \det{}'\,(d_0^{\dagger} d_0)^{1/2}\, \frac{1}{\mathrm{vol}\,(\ker d_0)}. \quad (31)$$

We have already seen that

$$\mathrm{vol}\,(\ker d_0) = \det\,(\phi_0^{\dagger} \phi_0)^{1/2}\ \mathrm{vol}\Big(\mathcal{H}^0(\mathrm{M})\Big). \quad (32)$$

Therefore the partition function of Maxwell theory is given (modulo a phase factor and with suitable normalization) by the same expression as (14).

To show that the Abelian Duality is a property of three dimensions only, consider again the de Rham complex. Due to Hodge duality, $d^{\dagger}_{m-p-1} = * d_p *$. Therefore $\det{}'\,(d^{\dagger}_{m-p-1} d_{m-p-1}) = \det{}'\,(d_p^{\dagger} d_p)$ and for even–dimensional manifolds all determinants involving the differential operator cancel each other. In higher odd dimensions, it is possible to find a relation between scalar field theory and Maxwell theory, but there will be more determinants coming in from the de Rham complex, thus non-physical theories should also be involved.

R $\longleftrightarrow$ 1/R duality can be shown in a different manner — by a duality transformation. We will illustrate this by considering the following partition function:

$$Z \;=\; \int\limits_{\ker d_1} \mathcal{D}A \;\; e^{\,R \int A \,\wedge\, * A} \;=\; \int\limits_{\ker d_1} \mathcal{D}A \;\; e^{\,R\,\langle A,\, A\rangle}. \tag{33}$$

Over the space of the kernel of the operator d_1 we can locally write $A = d_0\Phi$. Therefore Z becomes the partition function of free scalar field theory:

$$Z \;=\; \int\limits_{\Omega^0(M)} \mathcal{D}\Phi \;\; e^{\,R \int d_0\Phi \,\wedge\, * d_0\Phi}. \tag{34}$$

Alternatively, we can replace the integral over the kernel of the operator d_1 by an integral over $\Omega^1(M)$ and include a Lagrange multiplier B $\big(B \in \Omega^1(M)\big)$ to keep track of the fact that A is flat:

$$Z \;=\; \int\limits_{\ker d_1} \mathcal{D}A \;\; e^{\,R \int A \,\wedge\, * A} \;=\; \int\limits_{\Omega^1(M)} \mathcal{D}A \, \mathcal{D}B \;\; e^{\,R \int A \,\wedge\, * A \,+\, \int B \,\wedge\, d_1 A}. \tag{35}$$

If we integrate over A and absorb the resulting determinant det $(R\mathbb{1})$ in the normalization, we end up with the partition function of Maxwell theory with coupling constant $1/R$:

$$Z \;=\; \int\limits_{\Omega^1(M)} \mathcal{D}B \;\; e^{\,\frac{1}{R} \int d_1 B \,\wedge\, * d_1 B}. \tag{36}$$

The same can be seen if we make a change in the variables in (35) — *dualization* — $A \longrightarrow A' = A + \frac{1}{R} * d_1 B$.
With this dualization the partition function becomes:

$$Z \;=\; \int\limits_{\Omega^1(M)} \mathcal{D}A \;\; e^{\,R \int A \,\wedge\, * A} \int\limits_{\Omega^1(M)} \mathcal{D}B \;\; e^{\,\frac{1}{R} \int d_1 B \,\wedge\, * d_1 B}. \tag{37}$$

The integral over A is Gaussian and can be absorbed in the normalization factor. The remaining integral is the partition function of Maxwell theory.

Let us now include external currents J_μ in Maxwell theory and j in scalar field theory:

$$
\begin{aligned}
Z_1(J) &= \int\limits_{\Omega^1(M)} \mathcal{D}A \ e^{-i\int d^3x\,\sqrt{g}\,(F_{\mu\nu}F^{\mu\nu}+qJ_\mu A^\mu)} = \int\limits_{\Omega^1(M)} \mathcal{D}A \ e^{-i\langle A,\ d_1^\dagger d_1 A\rangle + q\,\langle J,\,A\rangle}, \\
Z_0(j) &= \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \ e^{-i\int d^3x\,\sqrt{g}\,(\partial_\mu\varphi\,\partial^\mu\varphi+ej\varphi)} = \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \ e^{-i\,\langle\varphi,\,d_0^\dagger d_0\varphi\rangle + e\,\langle j,\,\varphi\rangle}, \quad (38)
\end{aligned}
$$

where q and e are some charges.
Now extract perfect squares and perform the Gaussian integration to end up with:

$$
Z_0(j) \ = \ Z_0(J)\ \tau_{\mathrm{RS}}^{1/2}\,(\mathrm{M})\ exp\Big(-ie\,\langle j,\,\frac{1}{d_0^\dagger d_0}\,j\rangle\Big)\ exp\Big(iq\,\langle J,\,\frac{1}{d_1^\dagger d_1}\,J\rangle\Big). \quad (39)
$$

If the charges and the currents are related as:

$$
j \ = \ -i\sqrt{\frac{q}{e}}\,*\,d_2\,(d_1^\dagger)^{-1}\,J, \quad (40)
$$

then the Abelian duality will go through on the level of interactions with external currents.
Using the definition of correlation function (as a functional derivative of the partition function with respect to the external current), we can easily relate the correlation functions of scalar field theory, Maxwell theory and Chern–Simons theory.
We have recently shown [8] that the partition functions of Maxwell–Chern–Simons theory and the self-dual model differ by the partition function of Chern–Simons theory (thus the two theories being equivalent). Therefore, the ratio of the partition functions of scalar field theory and Maxwell theory is equal (modulo phase ambiguities) to the square of the ratio of the partition functions of Maxwell–Chern–Simons theory and the self-dual model. We can relate the correlation functions of these five models as well.
Finally we would like to mention that Chern–Simons theory can be dynamically generated from the parity–breaking part of a theory with massive fermions [9] — as gauge–invariant regularization of the massless fermionic determinant introduces parity anomaly given by the Chern–Simons theory. In this sense, the result of our paper (24) implies that a theory with massive fermions (interacting with external currents) together with massless scalar fields (possibly iteracting with external currents) add up to Maxwell theory (with possible interaction with external currents). Thus we have a form of bosonization in three dimensions.

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