Relativistic Shock Acceleration

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Abstract. In this paper we briefly review the basic theory of shock waves in relativistic hydrodynamics and magneto-hydrodynamics, emphasising some astrophysically interesting cases. We then present an overview of the theory of particle acceleration at such shocks describing the methods used to calculate the spectral indices of energetic particles. Recent results on acceleration at ultra-relativistic shocks are discussed

1 Introduction

Two recent observational developments have renewed interest in the subject of particle acceleration at relativistic shocks. The first concerns the strong variability of certain active galaxies in very high energy gamma-rays [1]. Secondly fireball models of gamma-ray bursts invoke the existence of relativistic shock waves to explain the burst and it's afterglow [2]. There has also been some speculation that acceleration at ultra relativistic shock fronts in fireballe models of GRBs may account for the ultra high energy cosmic rays - UHECRs ([3]). In light of these, and other theoretical, developments we present in this paper a review of the theory of particle acceleration at relativistic shocks. In section (2) we discuss hydrodynamic and magnetohydrodynamic shocks and particularly those factors (e.g. magnetic field strength in a relativistic shock) which influence the compression ratio which is the most important hydrodynamic quantity in the acceleration process. Energetic particle scattering and acceleration at relativistic shocks is described in section (3) while Monte Carlo simulations are covered in section (4) and a brief discussion of UHECR acceleration in ultra-relativistic shocks is contained in section (5). Much of the detail omitted from this short article is contained in a recent review [4] (henceforth referred to as KD99).

2 Relativistic shocks

The relativistic shock problem was solved by Taub [5] and, before turning to the MHD case, we will recall this theory. In the absence of external forces and energy sources, the

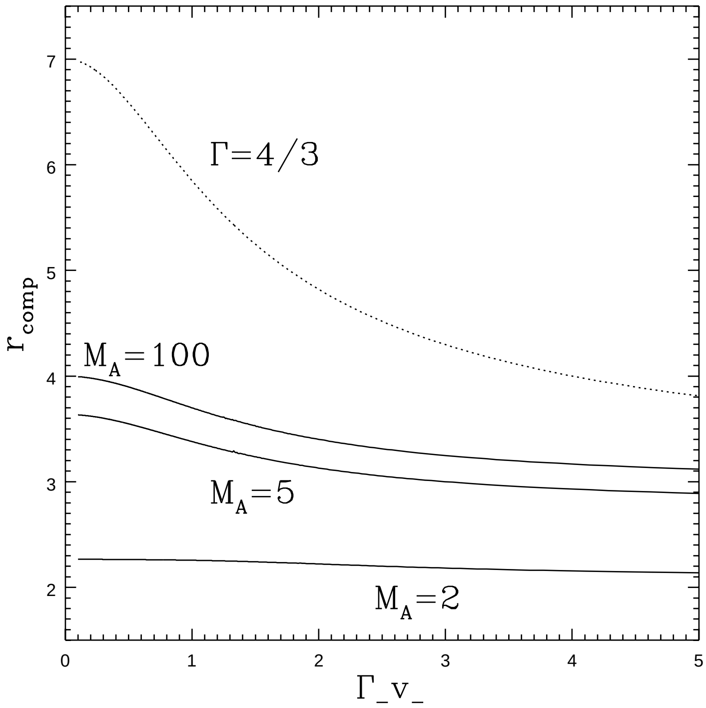

Fig. 1. The compression ratio of a strong oblique fast-mode shock front ($\Phi_- = 45^o$) propagating into a magnetised plasma. Various Alfvén Mach numbers $M_{\rm A}$ are shown, together with the hydrodynamic approximation (dotted line).

equations of relativistic hydrodynamics can be formulated as the vanishing divergence of the stress-energy tensor associated with the fluid:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1}$$

Neglecting dissipative effects, the stress-energy tensor is diagonal in the local plasma rest frame, and is given by $T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}$. Here u^μ is the four velocity of the fluid ($\mu = 0, 1, 2, 3$), and $g^{\mu\nu}$ the metric tensor, for which we adopt the convention $-{+}{+}{+}$. The scalars w and p are the proper enthalpy density and pressure, i.e., those measured in the rest frame of the fluid, in which $u^\mu = (1, 0, 0, 0)$. The problem of solving the Rankine-Hugoniot relations to find the jump conditions across a shock front requires one to use an equation of state to find the quantity p/ρ, given the quantity e/ρ. For a dissipation free ideal gas, the Synge equation is appropriate, and it is necessary to solve this numerically. A popular short-cut is to define a parameter $\hat\gamma$ via the equation $p = (\hat\gamma - 1)(e - \rho)$. In the non-relativistic case $\hat\gamma = 5/3$ and can be identified as the ratio of specific heats of the gas. For a gas whose pressure is dominated by a relativistic component, one has $\hat\gamma = 4/3$ (a fully relativistic gas has, in addition, $e \gg \rho$). Together with equation (1) the number conservation law,

$$\nabla_\mu (n_i u^\mu) \;=\; 0 \tag{2}$$

determines the jump conditions across a plane parallel shock. Defining the Lorentz scalars v_- and v_+ to be the shock speed measured in the upstream and downstream rest frames respectively and $\Gamma_\pm = (1 - v_\pm^2)^{-1/2}$ we have the conservation of mass, momentum and energy across the shock,

$$\Gamma_- \rho_- v_- \;=\; \Gamma_+ \rho_+ v_+ \tag{3}$$

$$\Gamma_-^2 w_- v_-^2 + p_- \;=\; \Gamma_+^2 w_+ v_+^2 + p_+ \tag{4}$$

$$\Gamma_-^2 w_- v_- \;=\; \Gamma_+^2 w_+ v_+ \tag{5}$$

Given v_- and the upstream state, e_-/ρ_-, these equations are to be solved for the downstream state v_+, e_+/ρ_- and the proper compression ratio $R \equiv \rho_+/\rho_-$. In general, this entails a numerical procedure, which is described in KD99 but it interesting to mention the special case of a shock in a relativistic gas in which $p = e/3$ (both upstream and downstream) where one finds the simple relation $v_- v_+ = 1/3$. Moreover, in the ultra-relativistic limit, $\Gamma_- \to \infty$, the upstream pressure (p_-) may be neglected in Eq. (4). If, in addition, the downstream particles are ultra-relativistic, in the sense that $e_+ \gg \rho_+$, one may combine Eqs. (4), (5) to find $v_+ \to \hat{\gamma} - 1 = 1/3$ and $\Gamma_{\rm rel} \to \Gamma_- \sqrt{(2-\hat{\gamma})/\hat{\gamma}} = \Gamma_-/\sqrt{2}$ These relations are independent of the equations of state upstream and downstream and hold whether or not particles are conserved at the shock, provided only that the downstream particles are ultra-relativistic.

In ideal, relativistic MHD, it is assumed that the plasma is dissipation free and that in the local rest frame the electric field vanishes. In this case, the electromagnetic field is specified by the magnetic field alone, so that the source-free Maxwell equations become

$$\nabla_\mu \left(B^\mu u^\nu - u^\mu B^\nu \right) = 0. \tag{6}$$

In the rest frame of the plasma $B^\mu = (0, \mathbf{B})$ where $\mathbf{B}$ is the magnetic field three vector in that frame. In the following, B is taken to denote the magnetic field strength in the local plasma rest frame which satisfies $B^\mu B_\mu = B^2$. The components of B^μ in a frame where the plasma is moving with four velocity u^μ can be derived from the appropriate Lorentz transformation in terms of $\mathbf{B}$. The energy momentum tensor of the system consisting of electromagnetic fields and fluid is

$$T^{\mu\nu} \;=\; \left(w + \frac{B^2}{4\pi} \right) u^\mu u^\nu + \left(p + \frac{B^2}{8\pi} \right) g^{\mu\nu} - \frac{B^\mu B^\nu}{4\pi}. \tag{7}$$

and the equations of relativistic MHD consist of the vanishing divergence of this tensor, together with (6) and, of course, an equation of state for the fluid. The jump equations follow from these conservation laws ([6], [7], [8]). As in the hydrodynamical case, there are some interesting limiting cases for MHD shocks. These include the cases of a weak dynamically unimportant magnetic field and ultra-relativistic perpendicular shocks (KD99). Of general importance to the acceleration problem is the fact that the compression which can be obtained in a relativistic shock decreases as the magnetic field becomes more and more important dynamically. This is illustrated in figure 1.

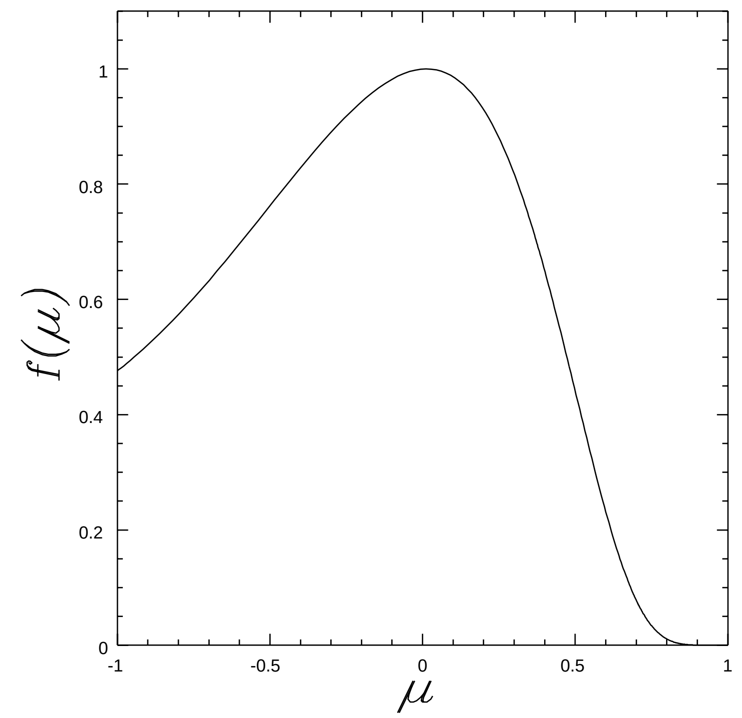

Fig. 2. The pitch-angle distribution of accelerated particles at a parallel relativistic shock front with $v_- = 0.9$, and $v_+ = 0.37$, as a function of the cosine μ of the pitch angle measured in the rest frame of the downstream plasma. Isotropic pitch-angle diffusion is assumed and the normalisation of f is arbitrary. The depletion of particles with $\mu \approx 1$, (those which move almost along the shock normal into the downstream plasma) arises because particles which move into the upstream plasma are overtaken again by the shock before undergoing substantial deflection [9].

3 Particle acceleration

In the absence of scattering, or if the particle mean free path is much smaller than the shock thickness, energetic particles will not undergo multiple crossings of the shock. They will, however, be compressed in passing from upstream to downstream. A simple model for such shock-drift acceleration is briefly presented in section 3.1. In the presence of pitch angle diffusion off MHD trubulence particles can cross the shock many times before being advected downstream . This first-order Fermi process produces a spectrum of energetic particles which, as in non-relativistic shocks, depends on the shock compression. However, the precise form of the pitch angle diffusion coefficient also plays a role in determining the spectral index. This is discussed in section 3.2.

3.1 Shock-drift acceleration

The dramatic increase in surface brightness which can be produced by a relativistic shock front merely as a result of the 'compression' of the electrons was pointed out by P. Scheuer [10]. A gas of relativistic electrons with an isotropic distribution function in the local fluid frame achieves isotropy by experiencing elastic scattering by slowly moving, low-frequency MHD waves, which may be self-excited. Consider energetic particles in a fluid element flowing into an MHD shock where the phase space density of particles is given by $f(p) = C_- p^{-s}$ between a lower and an upper cut off: $p_{\min} < p < p_{\max}$. When the scattering events are so rapid that the length scale over which they isotropise the electrons is much shorter than the length scale characterising the thickness of the shock, then the relativistic electrons react adiabatically. Particles are compressed according to $p\rho^{-1/3} = \text{constant}$ where ρ is the proper fluid density so that downstream of the shock front one has

$$\begin{aligned} f_+(p) &= f_-[p(\rho_+/\rho_-)^{1/3}] \\ &= C_- \left(\frac{\rho_+}{\rho_-}\right)^{s/3} p^{-s} \end{aligned} \tag{8}$$

In the absence of scattering, and with conservation of the particle's magnetic moment, the same qualitative result is found namely that although particles are accelerated in the shock-drift process the spectral index of the incoming distribution is unchanged.

3.2 First-order Fermi acceleration

In the case of test particles at a parallel shock, in the presence of pitch angle diffusion described by the coefficient $D_{\mu\mu}$, but neglecting as usual diffusion in energy, the equation to be solved is

$$\Gamma_\pm(1 + v_\pm \mu)\frac{\partial f}{\partial t} + \Gamma_\pm(v_\pm + \mu)\frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} D_{\mu\mu} \frac{\partial f}{\partial \mu} , \tag{9}$$

where we have assumed the Lorentz factor of the particles is much larger than that of the flow and have accordingly replaced their velocity with the speed of light ($= 1$). The relatively simple form of (9) is a consequence of a mixed coordinate system in which the cosine of the pitch angle μ is measured in the local rest frame of the plasma, but the space-time coordinates x and t refer to the rest frame of the shock front. Seeking stationary solutions one assumes a characteristic power law spectrum, $f \propto p^{-s}$ (where s is to be determined), and imposes the boundary conditions that the distribution vanishes far upstream while remaining finite at large distances downstream. There is a further, internal, boundary condition at the shock front itself which is situated at $x = 0$. Physically particles do not undergo a sudden change in momentum at the shock so that the distribution function remains continuous there,

$$f(p, \mu, x = 0) = \tilde{f}(\tilde{p}, \tilde{\mu}, x = 0) \tag{10}$$

where $(\tilde{p}, \tilde{\mu})$ are the momentum and pitch angle measured in the downstream frame which are related to those measured in the upstream frame (p, μ) by a Lorentz transformation. This describes all of the physics of particle acceleration but obtaining

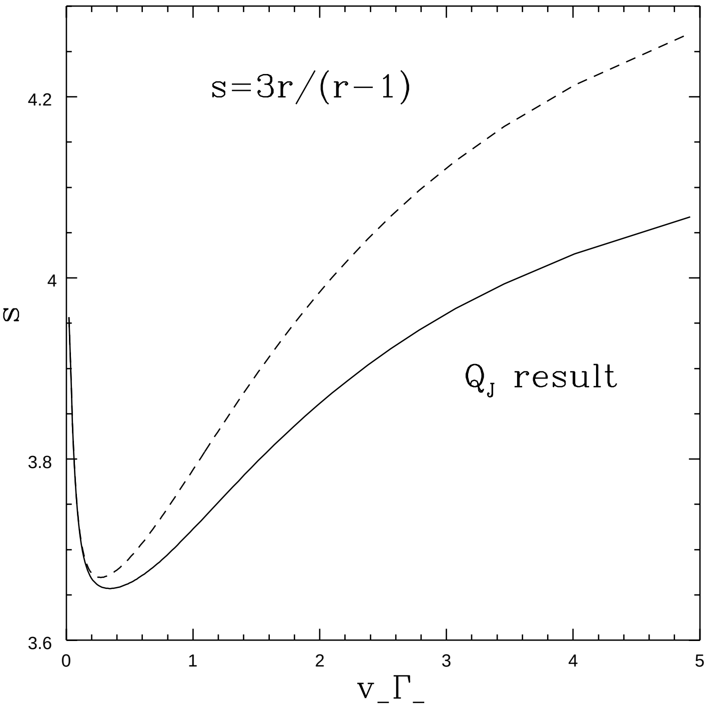

Fig. 3. The spectral index as determined using the Q_J method (solid line) for a parallel relativistic shock in plasma (25% helium) in full thermodynamic equilibrium. The dashed line corresponds to the non-relativistic formula for the spectral index: $s = 3r/(r-1)$, where $r = v_-/v_+$ is the compression ratio

the eigenvalues and eigenfunctions of the pitch angle distribution both upstream and downstream (i.e. actually solving 9 in the steady state) requires a numerical procedure. The technique for doing this, the Q_J method, is described in KD99.

An example of the angular distribution at a relativistic shock, as seen from the rest frame of the downstream plasma, is shown in Fig. 2. This figure was computed using an isotropic pitch-angle diffusion coefficient $D_{\mu\mu} \propto 1-\mu^2$. As well as confirming that the the distribution is strongly pitch-angle dependent, Fig. 2 shows that very few particles travel in the direction $\mu = 1$, i.e., along the shock normal into the downstream region. The reason is that a particle which crosses into the upstream plasma undergoes relatively little deflection before being caught again by the relativistically moving shock. Significant deviations from a naive extrapolation of the non-relativistic formula $s = 3r/(r-1)$ are found already at quite low speeds, as is shown in Fig. (3).

In contrast to the diffusive case, the value of the spectral index s for relativistic shocks depends on the functional form of the pitch-angle diffusion coefficient $D_{\mu\mu}$. This point has been investigated by Heavens and Drury [12] and by Kirk [11]. For

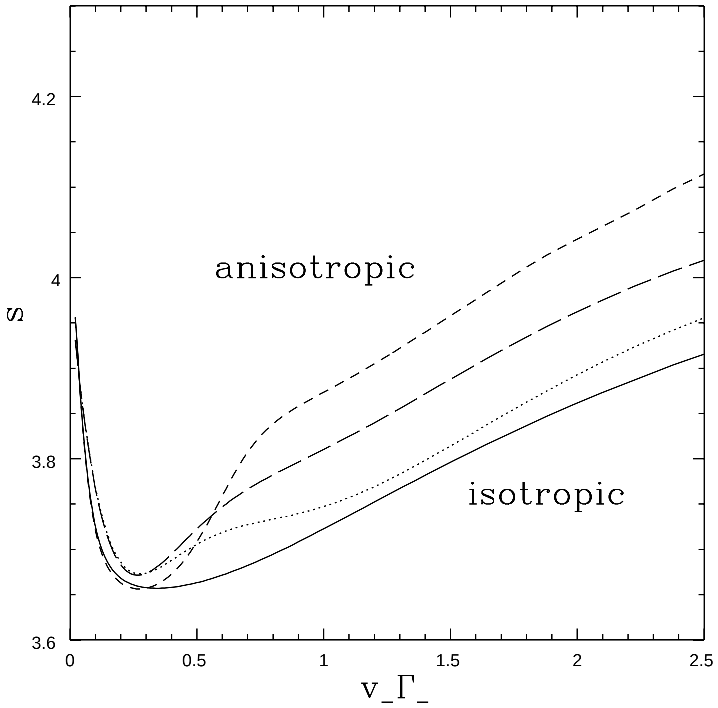

Fig. 4. The effect of anisotropic pitch angle diffusion on the spectral index produced by a parallel relativistic shock front. The full line (isotropic) and the short dashed line (anisotropic) show the results of the Q_J computation [11], with, for the latter, pitch angle diffusion given by equation (11). The spectral index found by Heavens and Drury [12] is shown by the dotted line (isotropic) and long dashed line (anisotropic). The pitch angle diffusion coefficient in the latter case is given by equation (12). Note that [11] and [12] assume slightly different compositions for the plasma.

example, if pitch-angle scattering through the point $\mu = 0$ is severely restricted, the spectrum is steepened. Figure 4 illustrates this for pitch-angle scattering given by

$$\begin{aligned} D_{\mu\mu} &\propto (1-\mu^2)\mu^q \qquad \text{for } |\mu| > \epsilon \\ D_{\mu\mu} &= \text{constant} \qquad \text{for } |\mu| < \epsilon \end{aligned} \tag{11}$$

with $\epsilon = 1/30$ and the index q, which corresponds to the power-law spectrum of the turbulent wave-energy in the quasi-linear theory equal, taken in this example to be 2. Heavens and Drury, on the other hand adopt the prescription

$$D_{\mu\mu} = (1-\mu^2)(\mu^2 + 0.01)^{1/3} \tag{12}$$

which roughly corresponds to the quasi-linear result in the presence of Kolmogorov turbulence.

To date, the method has been applied to shocks moving with a maximum Lorentz factor $\Gamma_- = 5$. Although the results in Fig. 4 seem to indicate a convergence to a value around 4.2 – a result also found and commented upon by Heavens and Drury [12] – there is no analytic guarantee that the asymptotic limit either exists or is approached smoothly. Notwithstanding this, recent numerical results ([13, 9] – see below) also find convergence to $s = 4.2$ for very large Lorentz factors.

4 Monte Carlo simulations

The basic idea of the Monte Carlo method is to find a way of constructing a stochastic trajectory whose distribution obeys the desired transport equation. Then, by repeating the procedure a large number of times, the distribution itself can be constructed approximately. The pitch-angle diffusion operator

$$\mathcal{C}[f(\mu)] \equiv \frac{\partial}{\partial\mu} D_{\mu\mu} \frac{\partial f}{\partial\mu} \tag{13}$$

results in a transport equation which is a second order differential equation, for example that given in equation (9). This describes the continuous deflection of the particle by an infinite succession of infinitesimally small changes in pitch angle (it is also possible to simulate large angle scattering which we consider to be qualitatively different to the continuous effect of turbulence on a particle's trajectory, KD99). The stochastic trajectory with such a collision operator is found by proceeding from one point on the trajectory (labelled by, say x_i, μ_i) to the next. One must solve the transport equation under the initial condition $f(x, \mu, t = 0) = \delta(x - x_i)\delta(\mu - \mu_i)$. For small changes in x, this can be done approximately by assuming that μ changes only slightly. Expanding equation (9) in powers of $\mu - \mu_i$, one finds

$$\Gamma_\pm(1 + v_\pm\mu_i)\frac{\partial f}{\partial t} + \Gamma_\pm(v_\pm + \mu_i)\frac{\partial f}{\partial x} = D_{\mu\mu}(\mu_i)\frac{\partial^2}{\partial\mu^2} f + D'_{\mu\mu}(\mu_i)\frac{\partial}{\partial\mu} f \tag{14}$$

where $D' = \mathrm{d}D/\mathrm{d}\mu$. The substitutions $\Delta = (t - t_i)/[\Gamma_\pm(1 + \mu_i v_\pm)]$, $\Xi = (x - x_i)/[\Gamma_\pm(v_\pm + \mu_i)]$ and $\eta = \mu - \mu_i + D'\Delta$ reduce this to the heat conduction equation, and the solution is easily seen to be

$$f[x, \mu, t = t_i + \Gamma_\pm(1 + \mu_i v_\pm)\Delta]) = \frac{\delta(x - x_i - \Gamma_\pm(v_\pm + \mu_i)\Delta)}{\sqrt{\pi D_{\mu\mu}\Delta}} \exp\left[-(\mu - \mu_i - D'_{\mu\mu}\Delta)^2/(D_{\mu\mu}\Delta)\right] \tag{15}$$

The next point on the trajectory is found by setting a sufficiently small time step Δ and choosing a new stochastic value of μ from the Gaussian distribution of equation (15). This method, together with specialised techniques for enhancing the statistical significance of the results (such as the 'splitting' technique) was applied to the the acceleration problem for particles undergoing synchrotron losses in [14].

Monte-Carlo simulations have also been performed for highly relativistic shocks by Bednarz & Ostrowski [13] for upstream Lorentz factors up to 240. They find the spectral index of accelerated particle converges to the value $s = 4.2$, independent of the orientation of the magnetic field, provided both pitch angle scattering and cross-field diffusion are permitted. In this ultra-relativistic limit, a particle which manages to cross a shock front from the downstream side into the upstream flow is very rapidly overtaken again once it is deflected. In fact it, can perform only a small fraction ($\sim 1/\Gamma_-$) of a gyration about the magnetic field line, unless the direction of the field is exactly along the shock normal [9]. In this case a combination of motion in a uniform field, and diffusion in angle due to fluctuations in the field on length scales much *shorter* than a gyro-radius arises. If the field fluctuates rapidly, one would expect to recover the operator equation (13), where now the quantity μ is interpreted not as the cosine of the pitch angle, but as the cosine of the angle between the particle velocity and the shock normal. Gallant et al [9] have extended the method to the ultra relativistic limit ($\Gamma_- \to \infty$) and considered both the case of diffusion in angle and scatter-free deflection by a uniform field. The corresponding power laws are $s = 4.25$ and 4.3 respectively, in reasonable agreement with Bednarz and Ostrowski, despite differences found in the angular distribution of the particles. This result is particularly encouraging for those theories of gamma-ray burst sources which use a relativistic blast wave to accelerate the particles, since it is close to the index of the particle spectrum required to produce afterglow spectra [15].

5 UHECRS and Ultra-Relativistic Shocks

There have been suggestions that ultra-relativistic shocks, particularly in the context of the fireball model of GRBs, might be capable of producing the UHECRs ([3]). Gallant and Achterberg [16] have studied this possibility by considering the first and subsequent shock crossing of a high energy particle at an ultra-relativistic shocks. Starting with an isotropic upstream distribution they find that a particle's energy is increased on average by a factor $\Gamma_{\rm rel}$ for the *first* shock crossing. However, for physically realistic deflection processes, it can be shown that for all *subsequent* crossings the energy is roughly doubled. While the shape of the spectrum is a power law with $f(p) \propto p^{-4.2}$ (as discussed above) the maximum energy is limited by the modest energy gains on all but the first cycle. Imposing the requirement that the time to deflect a particle upstream be less than the age of the fireball, Gallant and Achterberg find that the maximum energy attainable at the external shock of a fireball is,

$$E = 5 \times 10^{15}\, Z\, B_{-6}\, \eta_3^{1/3} \mathcal{E}_{52}^{1/3} n_0^{-1/3}\ \mathrm{eV}, \tag{16}$$

where Z is the particle's charge, $\mathcal{E} \equiv \mathcal{E}_{52}\, 10^{52}$erg is the fireball energy, and $B \equiv B_{-6}\, 10^{-6}$G and $n \equiv n_0\, \mathrm{cm}^{-3}$ are the surrounding medium magnetic field and density. This rules out the acceleration of UHECRs by repeated shock crossings at the external blast waves of GRB fireballs in galactic magnetic fields unless there are sufficently energetic particles upstream so that the initial large boost in energy is sufficient. Even in this instance one requires a medium composed predominantly of relativistic particles to reach the required efficiency for UHECR production. Gallant and Achterberg have therefore suggested that UHECRs might be produced by the initial boost in a relativistic fireball expanding into a pulsar wind bubble created by the progenitor system.

6 Conclusions

We have briefly reviewed the basic theory of relativistic shock acceleration and first reults which have now appeared for the first-order Fermi process at ultra-relativistic shocks. The compression ratio at an ultra-relativistic shock is independent of the shock speed. If, as the simulations suggest, the power-law index of accelerated particles turns out also to be independent of the shock speed in the ultra-relativistic limit, modelling of individual observational events will become simpler.

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