Discrete Approximation of the Riemann Problem for the Viscous Burgers Equation *

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Abstract. In this paper we consider discrete approximations of a Dirichlet problem for the quasilinear parabolic equation $L(u(x,t)) \equiv \{\varepsilon\partial^2/\partial x^2 - u(x,t)\partial/\partial x - \partial/\partial t - c(x,t)\}u(x,t) = f(x,t)$, that is, the viscous Burgers equation. The singular perturbation parameter ε takes arbitrary values from the half-interval (0,1]. The initial condition has a discontinuity of the first kind at the point $S^* = (0,0)$ such that $\varphi_0(+0) - \varphi_0(-0) > 0$, where $u(x,0) = \varphi_0(x)$; thus, we have the Riemann problem. For such a problem we construct special finite difference schemes controlled by the parameter ε and by the type of the singularities, which the solution $u(x,t)$ exhibits. The discrete solution for this problem is shown to converge uniformly with respect to the parameter ε in a uniform grid metric.

Introduction

The smoothness of solutions to singularly perturbed boundary value problems deteriorates as the singular perturbation parameter ε decreases. Reducing the smoothness of the boundary conditions also leads to a decreasing smoothness of the solution. The numerical solution of such problems gives rise to difficulties even in the linear case. Thus, the need arises for the development of special numerical methods whose solutions are convergent uniformly with respect to the parameter ε (see, for example, [1]–[6]).

Note that, as for the numerical approximation of linear boundary value problems for singularly perturbed parabolic equations, two approaches have generally been taken to construct special finite difference schemes: (a) an approach based on special difference approximations of differential operators on arbitrary (in particular, uniform) grids (the resulting schemes are called fitted operator schemes, see, e.g., [2], [3], [6]); (b) an approach that employs standard finite difference approximations of differential operators on special condensing grids, which guarantees the ε-uniform convergence of the difference scheme (these methods are called fitted mesh, or condensing mesh, methods, see, e.g., [1], [4]–[6]).

* This research was supported by the Russian Foundation for Basic Research under Grant 98-01-00362 and by the Enterprise Ireland grant SC-98-612.

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The present paper deals with the construction of grid approximations of singularly perturbed boundary value problem for a quasilinear parabolic equation. The initial condition has a discontinuity of the first kind. When the parameter equals zero, the parabolic equation reduces to a quasilinear hyperbolic one. The problem describes the decay of an initial discontinuity due to convection accompanied by slow diffusion (low viscosity) in the case when the rate of the convection is determined by the concentration of the substance (see, e.g., [7]–[9] and the references cited therein).

The construction of special schemes for linear singularly perturbed parabolic equations with a discontinuous initial condition was considered in [10]–[12]. Both approaches mentioned above were used in [10], [11] to construct the difference schemes. In [12], only the approach based on the fitted operator method was applied that considerably simplified the finite difference scheme.

Finite difference schemes that converge uniformly with respect to the parameter have been constructed in [13] for a quasilinear parabolic equation with a transient layer, where continuous boundary value conditions were considered. Special meshes condensing in a neighbourhood of the transient layer were used to construct these schemes.

In this work both the approaches are employed to construct the finite difference scheme. Special condensing meshes are used in a neighbourhood of the point of discontinuity. Fitted operators similar to those described in [12] are applied only in the nearest neighbourhood of the point of discontinuity. The distribution of the mesh points and also the appropriate difference approximations are determined by the value of the perturbation parameter ε and by the character of singularities of the exact solution.

1 Statement of the problem

On the interval

$$D_{(1.1)} = \{x: \ |x| < 1\} \tag{1.1}$$

we consider the Dirichlet problem for a quasilinear parabolic equation, with a small parameter multiplying the highest-order derivative

$$L_{(1.2)}\big(u(x,t)\big) = f(x,t), \qquad (x,t) \in G, \tag{1.2a}$$

$$u(x,t) = \varphi(x,t), \qquad (x,t) \in S. \tag{1.2b}$$

Here $G = D \times (0,T]$, $T > 0$, $S = S(G) = \overline{G} \setminus G$, $S = S_0 \cup S_1$, $S_0 = \{(x,t)\colon \ |x| \leq 1,$ $t = 0\}$ is the lower boundary of the set $\overline{G}$,

$$L_{(1.2)}\big(u(x,t)\big) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} - u(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t} - c(x,t)\right\} u(x,t),$$

$\varphi(x,t) = \varphi(x,0) \equiv \varphi_0(x)$ for $(x,t) \in S_0$. The parameter ε takes arbitrary values from the half-interval (0,1]. The coefficient $c(x,t)$ and the function $f(x,t)$ are sufficiently smooth on $\overline{G}$, and $c(x,t) \geq 0$, $(x,t) \in \overline{G}$. The boundary function $\varphi(x,t)$ has a discontinuity of the first kind at the point $S^\star$, $S^\star = (0,0)$. The function $\varphi(x,t)$ is sufficiently smooth outside of the set $S^\star$.

The notation $L_{(j.k)}$ ($D_{(j.k)}$, $f_{(j.k)}$, $M_{(j.k)}$) implies that this operator (set, function or constant) is introduced in the formula $(j.k)$.

By a solution of problem (1.2) is meant a function $u \in C(\overline{G} \setminus S^\star) \cap C^{2,1}(G)$ that satisfies equation (1.2a) on G supplemented with boundary condition (1.2b) on $S \setminus S^\star$.

The existence and uniqueness of solutions for regular quasilinear problems with continuous boundary conditions, as well as some *a priori* estimates, are considered in [14], [15].

We now define the data of problem (1.2), (1.1) more exactly. The boundary function $\varphi(x,t)$ satisfies the condition:

$$[\,\varphi_0(x)\,] > 0, \quad (x,t) \in S^\star, \tag{1.3}$$

and also the condition:

(A) the function $\varphi_0^*(x)$ is sufficiently smooth, and also $(d/dx)\varphi_0^*(x) \geq 0$, $x \in D$; here $\varphi_0^*(x) = \varphi_0(x) - [\varphi_0(0)]\chi(x)$, $\chi(x) = 2^{-1}\operatorname{sgn} x$, where $[v(x,t)]$ is the jump of the function $v(x,t)$: $[v(x,t)] = v(x+0,t) - v(x-0,t)$, $v(x \pm 0, t) = \lim\limits_{s \to 0,\, s>0} v(x \pm s, t)$.

When $\varepsilon \geq 0$ and $c(x,t) = f(x,t) \equiv 0$, equation (1.2a) reduces to the Burgers equation (the viscous Burgers equation for $\varepsilon > 0$). When $\varepsilon = 0$, equation (1.2a) is the hyperbolic equation

$$L_{(1.4)}\big(u(x,t)\big) = f(x,t), \quad (x,t) \in H, \tag{1.4a}$$

where $H = \overline{D} \times (0,T]$,

$$L_{(1.4)}\big(u(x,t)\big) \equiv \left\{-u(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t} - c(x,t)\right\} u(x,t),$$

with the initial condition

$$u(x,t) = \varphi_0(x), \quad (x,t) \in S_0^H, \tag{1.4b}$$

where $S_0^H = S_0^H(H) = \overline{H} \setminus H$.

We assume that the following conditions hold:

(B) the characteristics of the reduced equation (1.4a) (on the set $\overline{G}$), which respond to condition (1.4b), do not intersect with each other and leave the set $\overline{G}$ crossing its lateral boundary S_1 at nonzero angles;

(C) the solution of problem (1.2) is nondecreasing with respect to x.

The generalized solution of problem (1.4) is the limit of the solutions to problem (1.2) as $\varepsilon \to 0$. Under condition (1.3) the solution of the hyperbolic equation is continuous for

$t > 0$; thus, the discontinuity decays.

When applied to problems (1.2) and (1.4), the condition $[\,\varphi_0(x)\,] < 0,\ (x,t) \in S^\star$ leads to the appearance of an inner (transient) layer and to the discontinuity of the generalized solution, respectively, i.e., to the development of a shock wave.

The boundary value problem (1.2) is a model problem for compressible viscous gas flow in the case of a rarefaction wave [7]–[9].

We now discuss some difficulties which arise when the boundary value problem (1.2), (1.1) is solved by numerical methods.

The solution of this problem exhibits a rather complex behavior. In a small neighborhood of the point $S^\star$ as compared to the parameter ε, the solution of the boundary value problem is approximated, for $\varepsilon \in (0,1]$ and small values of t, by the solution of a singularly perturbed heat equation with a discontinuous initial condition. For small values of the parameter ε and at such distances from the point $S^\star$ that are essentially larger than the value of ε, the solution of the boundary value problem (1.2) is close to the solution of the reduced problem (1.4).

The derivatives of the solution to problem (1.2) tend to infinity as the point $S^\star$ is approached. When ε is sufficiently small, the derivatives of the solution vary sharply in a neighborhood of the singular rays that pass through the points $\{S^\star,\ u = \varphi_0(-0)\}$ and $\{S^\star,\ u = \varphi_0(+0)\}$. These singular rays are the characteristics of (1.4a). The first derivatives with respect to the space and time variables are of order t^{-1} between the singular rays (for $t \geq \varepsilon$). As $\varepsilon \to 0$, boundary layers appear in a neighborhood of the lateral boundary S_1.

The fact that the solution is nonsmooth at $S^\star$ and its derivatives are characterized by limited smoothness in the neighborhood of the singular rays, i.e., in weak transient layers, and also the presence of boundary layers lead to large errors when problem (1.2), (1.1) is solved by traditional numerical methods. Therefore, it is required to construct a finite difference scheme that converges uniformly with respect to the parameter (or, briefly, ε-uniformly) at each mesh point on the set $\overline{G}^\star = \overline{G} \setminus S^\star$.

Note that the construction of ε-uniformly convergent finite difference schemes for quasilinear equations with boundary layers was considered in [16]. In this paper, we are primarily interested in such special schemes which approximate the solution of boundary value problem (1.2) ε-uniformly in a neighborhood of the transient layers, namely, in a neighborhood of the point $S^\star$ and also on the set $\overline{G}$ for $|x| \leq 1 - m$, where m is sufficiently small number. For this reason, we assume for simplicity that the boundary conditions are chosen in such a way that no boundary layers appear, and among singularities of the boundary value problem are only weak transient layers as well as a discontinuity of the solution at the point S^*.

2 *A priori* estimates of the solution and its derivatives

In this section we give the estimates for the solution and its derivatives, which are used to construct and justify the difference schemes below.

For the solution of boundary value problem (1.2) we have the estimate

$$| \, u(x,t) \, | \leq M_{(2.1)}, \quad (x,t) \in \overline{G}, \tag{2.1}$$

where $M_{(2.1)} = T \max\limits_{\overline{G}} |f(x,t)| + \max\limits_{S} |\varphi(x,t)|$. This estimate is derived by using the maximum principle, cf [14], [15].

Here and below by M, M_i (m, m_i) we denote sufficiently large (sufficiently small) positive constants independent of the parameter ε. In discrete problems, these constants do not also depend on the disretisation parameters of the difference schemes used.

We complete a definition of the piecewise continuous function $v(x,t)$, $(x,t) \in S$, which is continuous on $S \setminus S^\star$, at the point of discontinuity $S^\star$ by the relation $v(x,t) = 2^{-1}(v(x+0,t) + v(x-0,t))$, $(x,t) \in S^\star$.

We introduce the auxiliary function $w_0(x,t)$ which is discontinuous on $S^\star$

$$w_0(x,t) = 2^{-1}\vartheta(2^{-1}\varepsilon^{-1/2}xt^{-1/2}), \quad (x,t) \in \overline{G} \setminus S^\star, \tag{2.2}$$

where

$$\vartheta(\xi) = \operatorname{erf} \xi = 2\pi^{-1/2} \int_0^\xi \exp(-\alpha^2) d\alpha$$

is the error function. The function $w_0(x,t)$ at $t > 0$ satisfies the homogeneous singularly perturbed heat equation

$$L_{(2.3)} w_0(x,t) \equiv \left\{ \varepsilon \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right\} w_0(x,t) = 0, \tag{2.3}$$

at $t = 0$ the function $w_0(x,t)$ is piecewise continuous, and also $[\, w_0(x,t) \,] = 1$, $(x,t) \in S^\star$.

Using the majorising function technique, we obtain the estimate

$$|u(x,t) - W_0(x,t) - \varphi_0^*(x)| \; \leq \; M\varepsilon^{-1/2}\, t^{1/2}, \quad (x,t) \in \overline{G}, \tag{2.4}$$

where

$$W_0(x,t) = [\, \varphi_0(0) \,] \, w_0(x,t). \tag{2.5}$$

Let $t_1 > 0$. On the set $G(t_1) = \{(x,t) : \; (x,t) \in G, \; t > t_1\}$, we consider the function $w_{(2.6)}(x,t)$ that approximates the solution of problem (1.2). We construct the function $w(x,t)$ to satisfy the initial value problem

$$\begin{aligned} L_{(1.2)}\Big(w(x,t)\Big) &= f(x,t), \qquad && (x,t) \in G(t_1), \\ w(x,t) &= W_0(x,t_1), && (x,t) \in S_0(t_1), \end{aligned} \tag{2.6}$$

where $S_0(t_1) = D \times [t = t_1]$. The function $w(x,t)$ is defined by its values on the boundary $S_1(t_1)$, where $S_1(t_1) = S(t_1) \setminus S_0(t_1)$, $S(t_1) = \overline{G}(t_1) \setminus G(t_1)$.

It is not difficult to construct $w(x,t)$ to satisfy the estimates

$$| u(x,t) - w(x,t) | \le M\varepsilon^{-1/2} t_1^{1/2}, \quad (x,t) \in \overline{G}(t_1), \tag{2.7a}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} w(x,t) \right| \le M \left[\varepsilon^{-k-2k_0} + (\varepsilon t_1)^{-k/2-k_0} \right], \quad (x,t) \in \overline{G}(t_1), \quad k + 2k_0 \le 4. \tag{2.7b}$$

In order to get estimate (2.7b), we pass to the variables $\xi = \varepsilon^{-1}x$, $\tau = \varepsilon^{-1}t$ in equation (2.6a) and condition (2.6b). Further, we use *a priori* estimates from [14], [15] to evaluate the function $\widetilde{v}(\xi,\tau) = w(x(\xi), t(\tau))$ and its derivatives.

Theorem 1 *Let the boundary function* $\varphi(x,t)$ *satisfy condition* (1.3), *and assume that conditions* (A), (B), (C) *are fulfilled. Then estimates* (2.1), (2.4) *and* (2.7) *hold for the solution of problem* (1.2).

3 *A priori* estimates based on asymptotics

We now give some *a priori* estimates related to the asymptotic behavior of the solution for small values of the parameter.

Let $\varphi^{(+)}(x)$ and $\varphi^{(-)}(x)$, $x \in R$ be sufficiently smooth functions that satisfy the relations: $\varphi^{(+)}(x) = \varphi_0(x)$, $0 < x < 1$; $\varphi^{(-)}(x) = \varphi_0(x)$, $-1 < x < 0$, and also $(d/dx)\varphi^{(\pm)}(x) \ge 0$, $x \in R$. We extend the functions $c(x,t)$ and $f(x,t)$ to the strip $\overline{H} = R \times [0,T]$ in a smooth way. We denote by $U^{(+)}(x,t)$ and $U^{(-)}(x,t)$ the solutions of the Cauchy problem for the hyperbolic equation

$$L_{(3.1)}\left(U^{(\pm)}(x,t)\right) \equiv \left\{ -U^{(\pm)}(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t} - c(x,t) \right\} U^{(\pm)}(x,t) = f(x,t),$$
$$(x,t) \in H, \qquad U^{(\pm)}(x,0) = \varphi^{(\pm)}(x), \quad x \in R. \tag{3.1}$$

If the functions $\varphi^{(+)}(x)$ and $\varphi^{(-)}(x)$ satisfy the condition $(d/dx)\varphi^{(\pm)}(x) > 0$ in a neighbourhood of the set $\overline{D}$, then for $t \le \delta_0$, where $\delta_0 > 0$ is a sufficiently small number, it follows that

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U^{(\pm)}(x,t) \right| \le M, \quad \frac{\partial}{\partial x} U^{(\pm)}(x,t) \ge 0, \quad (x,t) \in G, \quad t \in [0,\delta_0]. \tag{3.2}$$

The functions $U^{(+)}(x,t)$ and $U^{(-)}(x,t)$ are assumed to satisfy the conditions

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U^{(\pm)}(x,t) \right| \le M, \tag{3.3a}$$

$$\frac{\partial}{\partial x} U^{(\pm)}(x,t) \ge 0, \qquad (x,t) \in \overline{G}. \tag{3.3b}$$

In case condition (3.3) is true, the function $u(x,t)$ satisfies the estimates

$$\begin{aligned} \left|u(x,t) - U^{(+)}(x,t)\right| &\le M\left[\varepsilon + x^{-1}t\right], \quad x \ge M\varepsilon, \\ \left|u(x,t) - U^{(-)}(x,t)\right| &\le M\left[\varepsilon + |x|^{-1}t\right], \quad x \le -M\varepsilon, \quad (x,t) \in \overline{G}. \end{aligned} \tag{3.4}$$

The functions $U^{(+)}(x,t)$ and $U^{(-)}(x,t)$ approximate the solution of the boundary value problem, respectively, on the sets $x > 0$ and $x < 0$ for sufficiently small values of the parameter ε and the magnitude $|x|^{-1}t$.

Let $V(x,t),\ (x,t) \in \overline{H}$ be the solution of the Cauchy problem

$$L_{(3.5)}\Big(V(x,t)\Big) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} - V(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right\}V(x,t) = 0, \quad (x,t) \in H,$$

$$V(x,t) = \psi(x), \quad x \in R, \quad t = 0, \tag{3.5}$$

where

$$\psi(x) = \begin{cases} \varphi_0(+0), & x > 0, \\ \varphi_0(-0), & x < 0, \quad x \in R. \end{cases}$$

Taking into account estimates (3.4) and the explicit form of the function $V(x,t)$ [8], we establish the inequality

$$|u(x,t) - V(x,t)| \le M\left[\varepsilon + t^{1/2}\right], \quad (x,t) \in G, \quad |x| \le Mt^{1/2}. \tag{3.6}$$

When the values of t are small, the function $V(x,t)$ approximates the solution of the boundary value problem in the parabolic sector $t \ge mx^2$.

The following estimate holds for the function $V(x,t)$:

$$|V(x,t) - V_0(x,t)| \le M\varepsilon^{2/3}t^{-2/3}, \tag{3.7}$$

where

$$V_0(x,t) = \begin{cases} \varphi_0(-0), & x \le \varphi_0(-0)t, \\ xt^{-1}, & \varphi_0(-0)t < x \le \varphi_0(+0)t, \\ \varphi_0(+0), & \varphi_0(+0)t < x \end{cases}$$

is the solution of the Cauchy problem (3.5) for $\varepsilon = 0$:

$$L_{(3.8)}\Big(V_0(x,t)\Big) \equiv \left\{-V_0(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right\}V_0(x,t) = 0, \quad (x,t) \in H,$$

$$V_0(x,0) = \psi(x), \quad x \in R. \tag{3.8}$$

In the variables ξ, t, where $\xi = \xi(x,t) = xt^{-1}$, the function $\widetilde{V}(\xi,t) = V(x(\xi,t),t)$ obeys the equation

$$\begin{aligned} \widetilde{L}_{(3.9)}\left(\widetilde{V}(\xi,t)\right) \equiv \left\{\varepsilon t^{-2}\frac{\partial^2}{\partial \xi^2} - t^{-1}\left[\widetilde{V}(\xi,t) - \xi\right]\frac{\partial}{\partial \xi} - \frac{\partial}{\partial t}\right\}\widetilde{V}(\xi,t) = 0, \\ \xi \in R, \quad t > 0. \end{aligned} \tag{3.9}$$

The function $\widetilde{V}(\xi,t)$ satisfies the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial\xi^k\partial t^{k_0}}\widetilde{V}(\xi,t)\right| \le M(1+\xi+\varepsilon^{-1}t)^k(1+\xi+\varepsilon t^{-1})^{2k_0}, \qquad \xi\in R, \quad t>0;$$

$$\left|\frac{\partial^{k}}{\partial\xi^k}\widetilde{V}(\xi,t)\right| \le M\nu^{(1-k)/2}\exp\Big(-m\nu^{-1/2}\big(\xi-\varphi_0(+0)\big)\Big), \qquad \xi\ge\varphi_0(+0), \tag{3.10}$$

$$\left|\frac{\partial^{k}}{\partial\xi^k}\widetilde{V}(\xi,t)\right| \le M\nu^{(1-k)/2}\exp\Big(\ m\nu^{-1/2}\big(\xi-\varphi_0(-0)\big)\Big), \qquad \xi\le\varphi_0(-0),$$

$$\xi\in R, \quad t\ge M\varepsilon,$$

where $\nu=\varepsilon^{-1}t$, $m=m_{(3.10)}<1$, the constant $m_{(3.10)}$ can be chosen arbitrarily close to 1.

By virtue of estimates (3.4), (3.6) for $t_0>0$, where $t_0>0$ is an arbitrary number, a piecewise continuous function $\Phi(x,t_0)$ can be found such that

$$|\,u(x,t_0)-\Phi(x,t_0)\,| \le M\,t_0^{1/2}, \qquad \left|\frac{\partial}{\partial x}\Phi(x,t_0)\right| \le M\,t_0^{-1}, \qquad \frac{\partial}{\partial x}\Phi(x,t_0)\ \ge\ 0, \quad x\in D.$$

Let $\Phi_0(x,t)$ be a smooth function which approximates $\Phi(x,t)$ and satisfies the conditions

$$\Big|u(x,t_0)-\Phi_0(x,t_0)\Big| \le M\left[t_0^{1/2}+(\varepsilon^{1/2}+\delta_1)t_0^{-1}\right],$$

$$\left|\frac{\partial^k}{\partial x^k}\Phi_0(x,t_0)\right| \le Mt_0^{-1}(\varepsilon^{1/2}+\delta_1)^{1-k}, \quad \frac{\partial}{\partial x}\Phi_0(x,t_0)>0, \quad x\in D, \quad k=1,2,3.$$

Here $\delta_1\ge 0$ is a sufficiently small value. Then, for $t\in[\,t_0,t_0+\delta_0\,]$, where $\delta_0>0$ is also sufficiently small, the solution of the problem

$$\begin{aligned} L_{(3.1)}\big(Z(x,t)\big) &= f(x,t), \quad (x,t)\in G, \quad t\in(t_0,t_0+\delta_0\,], \\ Z(x,t_0) &= \Phi_0(x,t_0), \end{aligned} \tag{3.11}$$

exists and satisfies the relations

$$|u(x,t)-Z(x,t)| \le M\,[\,t_0^{1/2}+(\varepsilon^{1/2}+\delta_1)t_0^{-1}\,], \tag{3.12a}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}Z(x,t)\right| \le M\,t_0^{-1}(\varepsilon^{1/2}+\delta_1)^{1-k-k_0}, \tag{3.12b}$$

$$\frac{\partial}{\partial x}Z(x,t)\ \ge\ 0, \quad (x,t)\in G, \quad t\in[\,t_0,t_0+\delta_0\,]. \tag{3.12c}$$

The estimate (3.12a) for $t_0=\varepsilon^{1/3}$, $\delta_1=0$ implies the inequality

$$|u(x,t)-Z(x,t)| \le M\,\varepsilon^{1/6}, \quad (x,t)\in\overline{G}, \quad t\in[\,\varepsilon^{1/3},\varepsilon^{1/3}+\delta_0\,],$$

where, generally speaking, $\delta_0=\delta_0(\varepsilon)$. The function $u_0(x,t)$ defined by

$$u_0(x,t)=\begin{cases} U^{(-)}(x,t), & x<-M\,t^{1/2}, \\ U^{(+)}(x,t), & x>M\,t^{1/2}, \\ V(x,t), & |x|\le M\,t^{1/2}, \quad t\le\varepsilon^{1/3}; \\ Z(x,t), & x\in D, \qquad\quad t\ge\varepsilon^{1/3}, \end{cases} \tag{3.13}$$

is the main term in the asymptotic expansion of the solution $u(x,t)$ of the boundary value problem (1.2):

$$|u(x,t) - u_0(x,t)| \le M\,\varepsilon^{1/6}, \quad (x,t) \in \overline{G}, \quad t \le \varepsilon^{1/3} + \delta_0. \tag{3.14}$$

Theorem 2 *Let the assumptions of Theorem* 1 *hold. Then the auxiliary functions* $U^{(+)}(x,t)$, $U^{(-)}(x,t)$, $V(x,t)$ *and* $Z(x,t)$, *which are the solutions of problem* (3.1), (3.5) *and* (3.11), *satisfy the estimates* (3.2), (3.7), (3.10) *and* (3.12b), (3.12c), *respectively. For the function* $u(x,t)$, *i.e., the solution of boundary value problem* (1.2), *the estimates* (3.4), (3.6), (3.12a), (3.14) *are valid.*

Remark. The estimates (3.4), (3.6) are valid for $t \in [\,t_0, T\,]$. Assume the relations (3.12b), (3.12c) are fulfilled for $t \in [\,t_0, T\,]$. Then, for $t \in [\,t_0, T\,]$, the estimates (3.12a), (3.14) hold. It is not difficult to satisfy the relations (3.12b), (3.12c) for $t \in [\,t_0, T\,]$, for example, with $c(x,t)$, $f(x,t) \equiv 0$ (see also discussions in the subsection 3 of Section 7).

4 Difference scheme convergent for bounded values of ε

When constructing the scheme, we assume that the estimates of Theorem 1 hold.

On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0. \tag{4.1a}$$

Here $\overline{\omega}$ and $\overline{\omega}_0$ are uniform grids on the intervals $[-1,1]$ and $[0,T]$, respectively. We denote by h and h_0 the steps of the grids $\overline{\omega}$ and $\overline{\omega}_0$, respectively, $h = 2N^{-1}$, $h_0 = TN_0^{-1}$, where N and N_0 are the numbers of intervals in the grids $\overline{\omega}$ and $\overline{\omega}_0$, respectively. Assume $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$. On the set $S_h^\star = S^\star \cap \overline{G}_h$, the boundary function $\varphi(x,t)$ is defined by $\varphi(x,t) = 2^{-1}(\varphi(x+0,t) + \varphi(x-0,t))$, $(x,t) \in S_h^\star$. We represent the domain G as a sum of the subdomains

$$G = G_1 \cup G_2, \tag{4.1b}$$

where $G_1 = \{(x,t):\ x \in D,\ 0 < t \le t_1\}$, $G_2 = G \setminus G_1$, $t_1 = \min\{\varepsilon^{-3/5}[N^{-1} + N_0^{-1}]^{2/5}, T\}$. On the set G_2, to approximate equation (1.2a) we use the classical finite difference scheme

$$\Lambda_{(4.2)}(z(x,t)) = f(x,t), \qquad (x,t) \in G_{2h}, \tag{4.2}$$

where

$$\Lambda_{(4.2)}\big(z(x,t)\big) = \Big\{\, \varepsilon\delta_{x\overline{x}} - z^+(x,t)\delta_{\overline{x}} - z^-(x,t)\delta_x - \delta_{\overline{t}} - c(x,t) \,\Big\}\, z(x,t),$$

$z^+(x,t) = z(x,t) + |z(x,t)|$, $z^-(x,t) = z(x,t) - |z(x,t)|$, $\delta_{\overline{t}}\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$, $\delta_{x\overline{x}}\, z(x,t)$ are the classical first and second difference derivatives on uniform grids $\overline{\omega}_0$ and $\overline{\omega}$, respectively,

$$G_{sh} = G_s \cap \overline{G}_h, \quad s = 1,\, 2, \quad \overline{G}_h = \overline{G}_{h(4.1a)}. \tag{4.1c}$$

On the set G_1, we approximate equation (1.2a) by the fitted finite difference scheme

$$\Lambda_{(4.3)}\Big(z(x,t)\Big) = f(x,t), \qquad (x,t) \in G_{1h}, \tag{4.3a}$$

where

$$\Lambda_{(4.3)}\Big(z(x,t)\Big) = \Big\{ \varepsilon\gamma(x,t)\delta_{x\overline{x}} - z^+(x,t)\delta_{\overline{x}} - z^-(x,t)\delta_x - \delta_{\overline{t}} - c(x,t) \Big\} z(x,t).$$

The fitting coefficient $\gamma(x,t)$ is chosen according to the fitted operator principle [2], [3] so that the function $w_0(x,t) = w_{0(2.2)}(x,t)$ must be the solution of the homogeneous difference equation which approximates the singularly perturbed heat equation

$$\Lambda_{(4.4)} w_0(x,t) \equiv \{\varepsilon\gamma(x,t)\delta_{x\overline{x}} - \delta_{\overline{t}}\} w_0(x,t) = 0, \qquad (x,t) \in G_h. \tag{4.4}$$

It is convenient to choose the function $\gamma(x,t)$ such that the equation $\Lambda_{(4.4)}v(x,t) = 0$, $(x,t) \in G_h$ is satisfied with the function $v(x,t) = w_0(x,t) + u_0(x,t)$, where $u_0(x,t)$ is any smooth solution of the homogeneous equation $L_{(2.3)}u(x,t) = 0$, $(x,t) \in G$, for example,

$$u_0(x,t) = u_{0(4.5)}(x,t) \equiv -x^3 - 6\varepsilon xt, \quad (x,t) \in \overline{G}. \tag{4.5}$$

Then

$$\gamma(x,t) = \frac{\delta_{\overline{t}}\, w_0(x,t) + \delta_{\overline{t}}\, u_0(x,t)}{\varepsilon\delta_{x\overline{x}}\, w_0(x,t) + \varepsilon\delta_{x\overline{x}}\, u_0(x,t)}, \qquad x \neq 0. \tag{4.3b}$$

Assume

$$\gamma(x,t) = 1 \quad \mathit{for} \quad x = 0. \tag{4.3c}$$

Note that, for $u_0(x,t) \equiv 0$, the function $\gamma(x,t)$ is very sensitive to computing errors, because of the quick exponential convergence of $\delta_{x\overline{x}}\, w_0(x,t)$ to zero as the value of $\varepsilon^{-1/2}xt^{-1/2}$ increases.

To solve problem (1.2), we use the difference scheme

$$\begin{aligned} \Lambda_{(4.6)}(z(x,t)) &= f(x,t), \qquad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \qquad (x,t) \in S_h. \end{aligned} \tag{4.6}$$

Here

$$\Lambda_{(4.6)}(z(x,t)) = \begin{cases} \Lambda_{(4.3)}(z(x,t)), & (x,t) \in G_{1h}, \\ \Lambda_{(4.2)}(z(x,t)), & (x,t) \in G_{2h}. \end{cases}$$

The quasilinear difference scheme (4.6), (4.1) is monotone, and, just as for problem (1.2), the maximum principle [17] holds.

We assume that, for the considered data of problem (1.2), the solution of the discrete problem exists and satisfies a condition similar to (C):

(D) the discrete solution is nondecreasing with respect to the variable x.

When evaluating the error of the solution $z(x,t) - u(x,t)$, we use the majorant function technique (see, e.g., [16]). Taking into account the estimates for $\gamma(x,t)$, $\gamma(x,t) - 1$, and also the *a priori* estimates for the solution of the boundary value problem, we find

$$|u(x,t) - z(x,t)| \le M\left\{N_0^{-3/2}N + \varepsilon^{-4/5}[\,N^{-1} + N_0^{-1}\,]^{1/5}\right\}, \quad (x,t) \in \overline{G}_h.$$

Under the condition

$$N_0^{-13/10} \;\le\; \mathcal{O}(N^{-1}) \tag{4.7}$$

we have

$$|u(x,t) - z(x,t)| \;\le\; M\varepsilon^{-4/5}\left[N^{-1} + N_0^{-1}\right]^{1/5}, \quad (x,t) \in \overline{G}_h. \tag{4.8}$$

Thus, in the case of condition (4.7), the difference scheme (4.6), (4.1) converges for

$$\varepsilon^{-4/5}\left[N^{-1} + N_0^{-1}\right]^{1/5} \to 0, \tag{4.9}$$

in particular, for the fixed finite value of the parameter ε.

Theorem 3 *Assume that conditions* (A)–(D) *are fulfilled, and let the estimates of Theorem* 1 *hold for the solution of the boundary value problem* (1.2)*. Then, under conditions* (4.7) *and* (4.9)*, the solution of the finite difference scheme* (4.6)*,* (4.1) *converges to the solution of problem* (1.2)*. The discrete solution satisfies* (4.8)*.*

5 Difference scheme convergent for small values of ε

In this section, for small values of the parameter ε, we construct a difference scheme using the domain decomposition method for boundary value problem (1.2).

1. First we introduce the domain decomposition method. We cover the domain G by a system of subdomains, in each of them we construct a suitable grid approximation for problem (1.2). The size and form of these subdomains depend on the value of the parameter ε and the number of mesh points in the space and time grids, which are used for constructing the mesh discretisations in the subdomains. Assume

$$G = \bigcup_{i=1}^{4} G_i, \tag{5.1a}$$

where $G_1 = G \cap \{0 < t \le t_1\}$, $G_i = G \cap \{t_{i-1} < t \le t_i\}$, $i = 2,3,4$, $0 < t_i \le t_k \le t_4 = T$, $i < k$. We represent each set G_i, $i = 1,2,3$ as a sum of the subdomains:

$$G_i = \bigcup_{j=1}^{3} G_i^j, \quad i = 1,2,3, \tag{5.1b}$$

where

$$G_1^1 = G_1 \cap \{|x| < 2l_1\}, \quad G_1^2 = G_1 \cap \{x < -l_1\}, \quad G_1^3 = G_1 \cap \{x > l_1\},$$

$$G_2^1 = G_2 \cap \{|x| < 2æ_2 t\}, \quad G_2^2 = G_2 \cap \{x < -æ_2 t\}, \quad G_2^3 = G_2 \cap \{x > æ_2 t\},$$

$$G_3^1 = G_3 \cap \{|x| < 2æ_3 t\}, \quad G_3^2 = G_3 \cap \{x < -æ_3 t\}, \quad G_3^3 = G_3 \cap \{x > æ_3 t\},$$

$æ_2 = l_1 t_1^{-1}$, $æ_3 = l_2 t_2^{-1}$, $0 < l_1,\ l_2 \le 1$; the values of l_1, l_2, t_1, t_2, t_3 will be defined below. For convenience in the sequel we introduce a new numbering of the subdomains. Assume

$$G^k = G_i^j, \quad where \quad k = 3(i-1)+j; \quad k = 1, 2, \ldots, 10, \tag{5.1c}$$

here $G_4 = G_4^1$.

We now construct the alternating Schwartz method for the differential problem. On the subdomains G^4, G^7 we pass from the variables x, t to the new variables ξ, t, where $\xi = \xi(x,t) = xt^{-1}$. In terms of ξ and t, equation (1.2a) takes the form

$$L_{(5.2)}(\tilde{u}(\xi,t)) = \tilde{f}(\xi,t), \quad (\xi,t) \in \tilde{G}, \tag{5.2}$$

where

$$L_{(5.2)}(\tilde{u}(\xi,t)) \equiv \left\{ \varepsilon t^{-2} \frac{\partial^2}{\partial \xi^2} - t^{-1}(\tilde{u}(\xi,t) - \xi) \frac{\partial}{\partial \xi} - \frac{\partial}{\partial t} - \tilde{c}(\xi,t) \right\} \tilde{u}(\xi,t),$$

$$\tilde{v}(\xi,t) = v(x(\xi,t),t), \quad \tilde{G}^0 = \left\{ (\xi,t) :\ \xi = \xi(x,t),\ (x,t) \in G^0 \right\}, \quad x(\xi,t) = \xi t,$$

$\tilde{G}^0$ is an image of the set G^0, G^0 is a subset of the set $\overline{G}$, and $v(x,t)$ is any one of the functions $u(x,t)$, $c(x,t)$, $f(x,t)$.

Let $u^0(x,t)$, $(x,t) \in \overline{G}$ be an arbitrary function satisfing condition (1.2b). Assume

$$\begin{aligned} L_{(5.3)}(u^{r+\frac{1}{K}}(x,t)) &= 0, && (x,t) \in G^1, \\ u^{r+\frac{1}{K}}(x,t) &= u^r(x,t), && (x,t) \in \overline{G} \setminus G^1; \end{aligned} \tag{5.3}$$

$$\begin{aligned} L_{(5.3)}(u^{r+\frac{k}{K}}(x,t)) &= 0, && (x,t) \in G^k, && k \ne 1, 4, 7, K, \\ L_{(5.3)}(\tilde{u}^{r+\frac{k}{K}}(\xi,t)) &= 0, && (\xi,t) \in \tilde{G}^k, && k = 4, 7, \\ u^{r+\frac{k}{K}}(x,t) &= u^{r+\frac{k-1}{K}}(x,t), && (x,t) \in \overline{G} \setminus G^k, && k = 2,\ \ldots,\ K-1; \\ L_{(5.3)}(u^{r+1}(x,t)) &= 0, && (x,t) \in G^K, && \\ u^{r+1}(x,t) &= u^{r+\frac{K-1}{K}}(x,t), && (x,t) \in \overline{G} \setminus G^K; && r = 0, 1, 2, \ldots. \end{aligned}$$

Here $K = 10$,

$$L_{(5.3)}(u(x,t)) \equiv L_{(1.2)}(u(x,t)) - f(x,t), \quad (x,t) \in G^k,$$

$$k = 1, \ldots, K, \quad k \ne 4, 7;$$

$$L_{(5.3)}(\tilde{u}(\xi,t)) \equiv L_{(5.2)}(\tilde{u}(\xi,t)) - \tilde{f}(\xi,t), \quad (\xi,t) \in \tilde{G}^k \quad for \quad k = 4, 7;$$

$$u^{r+\frac{k}{K}}(x,t) = \tilde{u}^{r+\frac{k}{K}}(\xi(x,t),t) \quad for \quad k = 4, 7;$$

r is the number of iteration, $r = 0, 1, 2, \ldots$. Each function $u^{r+\frac{k}{K}}(x,t)$, $(x,t) \in \overline{G}$ is the solution of the Dirichlet problem on the set $\overline{G}^k$:

$$\begin{aligned} L_{(5.3)}(u^{r+\frac{k}{K}}(x,t)) &= 0, && (x,t) \in G^k, && \\ u^{r+\frac{k}{K}}(x,t) &= u^{r+\frac{k-1}{K}}(x,t), && (x,t) \in S^k, \quad k = 1, \ldots, K, && k \ne 4, 7; \\ L_{(5.3)}(\tilde{u}^{r+\frac{k}{K}}(\xi,t)) &= 0, && (\xi,t) \in \tilde{G}^k, && \\ \tilde{u}^{r+\frac{k}{K}}(\xi,t) &= \tilde{u}^{r+\frac{k-1}{K}}(\xi,t), && (\xi,t) \in \tilde{S}^k, && k = 4, 7 \end{aligned}$$

and coincides with the function $u^{r+\frac{k-1}{K}}(x,t)$, $(x,t) \in \overline{G}$ on the set $\overline{G} \setminus G^k$; here $S^k = \overline{G}^k \setminus G^k$. It is required to find the sequence of functions $u^r(x,t)$, $(x,t) \in \overline{G}$, $r = 1, 2, \ldots$, which are components of the solution to problem (5.3), (5.1). This is the alternating Schwartz method.

Using the comparison theorems [14], [15], we establish the estimate

$$|u(x,t) - u^r(x,t)| \leq Q(\varepsilon, \delta) \, (q(\varepsilon, \delta))^r, \quad (x,t) \in \overline{G}. \tag{5.4}$$

Here

$$q(\varepsilon, \delta) < 1, \quad \delta = \min_{i,\,(x^1,t),\,(x^2,t)} \rho_i\left((x^1,t),\,(x^2,t)\right), \quad (x^1,t) \in \overline{G}_i^j, \quad (x^2,t) \in \overline{G}_i^{[j]},$$
$$(x^1,t),\ (x^2,t) \notin \left\{ G_i^j \bigcap G_i^{[j]} \right\}, \quad i,\, j = 1, 2, 3,$$

$G_i^{[j]}$ is the union of the subdomains G_i^1, G_i^2, G_i^3 excluding the set G_i^j

$$G_i^{[j]} = \bigcup_k G_i^k, \quad k = 1,\ 2,\ 3, \quad k \neq j,$$

$\rho_i((x^1,t),(x^2,t))$ is the distance between the points (x^1,t) and (x^2,t) from the set $G_{i(5.1)}$.

We consider such a method for solving problem (1.2) as the domain decomposition method in the case of nonempty intersection of the subsets. It is required to find the functions $u^k(x,t)$, $(x,t) \in \overline{G}$, $k = 1, \ldots, K$ $(K = 10)$ which satisfy the relations

$$L_{(5.3)}(u^k(x,t)) = 0, \qquad (x,t) \in G^k, \quad k = 1, \ldots, K, \quad k \neq 4, 7, \tag{5.5a}$$

$$L_{(5.3)}(\tilde{u}^k(\xi,t)) = 0, \qquad (\xi,t) \in \tilde{G}^k, \quad k = 4, 7, \tag{5.5b}$$

$$u^k(x,t) = u^{k-1}(x,t), \qquad (x,t) \in \overline{G} \setminus G^k, \quad k = 1, \ldots, K, \tag{5.5c}$$

where $u^{k-1}(x,t) = u^K(x,t)$ for $k = 1$.

The following relation holds for the domain decomposition method (5.5), (5.1):

$$u^k(x,t) = u(x,t), \quad (x,t) \in \overline{G}, \quad k = 1, \ldots, K, \tag{5.6}$$

where $u(x,t)$ is the solution of problem (1.2). In (5.5), (5.6) the upper index k shows the number of that subdomain $\overline{G}^k$ where the auxiliary problem is considered.

Taking into account relations (5.4), (5.6), the alternating Schwartz method (5.3), (5.1) for solving problem (1.2), (1.1) can be interpreted as an iterative method for the solution of problem (5.5), (5.1) (that is, the domain decomposition method for problem (1.2), (1.1)).

2. In this subsection we construct mesh approximations for the domain decomposition method (5.5), (5.1).

On the chosen domains $G^k_{(5.1)}$ we construct the grids. For simplicity, when constructing the grids in each of the subdomains (if it is nonempty), we use $N + 1$ and $N_0 + 1$ nodes with respect to the space and time variables, respectively. We define the values of t_1, t_2, t_3, l_1, l_2, assuming

$$t_1 = \varepsilon N_\star^{-10/63}, \quad t_2 = \min\left[\varepsilon N_\star^{2/7}, T\right], \quad t_3 = \max\left[\varepsilon^{1/3} + N_\star^{-1/3}, t_2\right],$$
$$l_1 = \varepsilon N_\star^{-1/63}, \quad l_2 = 2\,t_2\,(1 + \max\{\,|\varphi_0(-0)|, |\varphi_0(+0)|\,\}), \tag{5.7}$$

where $N_\star = \min[N, N_0]$. Note that the domains G^k, $k = 5, 6, 8, 9$ are not rectangles. We construct the grids G_h^k, $k = 1, \dots, 10$, $k \neq 4, 7$ and $\widetilde{G}_h^k$, $k = 4, 7$ on the basis of rectangular grids, as follows:

$$G_h^k = G^k \bigcap \left\{\overline{\omega}^k \times \overline{\omega}_0^k\right\}, \quad k = 1, \dots, 10, \quad k \neq 4, 7,$$
$$\widetilde{G}_h^k = \widetilde{G}^k \bigcap \left\{\overline{\widetilde{\omega}}^k \times \overline{\omega}_0^k\right\}, \quad k = 4, 7. \tag{5.8a}$$

All the grids $\overline{\omega}^k$, $\overline{\widetilde{\omega}}^k$, $\overline{\omega}_0^k$, besides the grid $\overline{\widetilde{\omega}}^7$, are uniform grids. The distribution of the nodes for the mesh $\overline{\widetilde{\omega}}^7 = \overline{\widetilde{\omega}}^7(\sigma)$ is piecewise-uniform: the nodes of $\overline{\widetilde{\omega}}^7(\sigma)$ are condensed in a neighborhood of the points $\xi = \xi_1, \xi_2$, where $\xi_1 = \varphi_0(-0)$ and $\xi_2 = \varphi_0(+0)$; σ is a parameter depending on $N_\star$ and N. The mesh $\overline{\widetilde{\omega}}^7(\sigma)$ is constructed to be uniform on each of the intervals $[\,-\xi_0, \xi_1 - \sigma\,]$, $[\,\xi_1 - \sigma, \xi_1 + \sigma\,]$, $[\,\xi_1 + \sigma, \xi_2 - \sigma\,]$, $[\,\xi_2 - \sigma, \xi_2 + \sigma\,]$, $[\,\xi_2 + \sigma, \xi_0\,]$, where $\xi_0 = 4\big(1 + \max\{|\varphi_0(-0)|, |\varphi_0(+0)|\,\}\big)$; the number of nodes in each of these intervals is $5^{-1}N + 1$. The value of σ is given by the relation

$$\sigma = \sigma^7 = \sigma(N_\star, N) = \min\left[4^{-1},\ 4^{-1}(\varphi_0(+0) - \varphi_0(-0)),\ N_\star^{-1/2}\ln N\right]. \tag{5.8b}$$

On the boundaries S^k and $\widetilde{S}^k$, the meshes (5.8a) define the sets of the boundary nodes S_h^k and $\widetilde{S}_h^k$ (see, e.g., [17]) in a natural way. The sets S_h^k and $\widetilde{S}_h^k$ consist of the points which are formed by intersection of the lines $t = t^s$, $t^s \in \omega_0^k$ with the boundaries S^k and $\widetilde{S}^k$, respectively, and also the points $\overline{\omega}^k \times \{\overline{\omega}_0^k \setminus \omega_0^k\}$ and $\overline{\widetilde{\omega}}^k \times \{\overline{\omega}_0^k \setminus \omega_0^k\}$, which belong to the lower bases of the sets $\overline{G}^k$ and $\overline{\widetilde{G}}^k$. Assume

$$\overline{G}_h^k = G_h^k \bigcup S_h^k, \quad \overline{\widetilde{G}}_h^k = \widetilde{G}_h^k \bigcup \widetilde{S}_h^k. \tag{5.8c}$$

On the grids $\overline{G}_h^k$ and $\overline{\widetilde{G}}_h^k$ it is required to find the grid functions $z^k(x,t)$ and $\widetilde{z}^k(\xi,t)$. On the sets G_h^k and $\widetilde{G}_h^k$, these functions satisfy the following discrete equations which approximate equations (5.5a) and (5.5b)

$$\Lambda_{(5.9)}(z^k(x,t)) = 0, \quad (x,t) \in G_h^k, \quad k = 1, \dots, K, \quad k \neq 4, 7, \tag{5.9a}$$
$$\Lambda_{(5.9)}(\widetilde{z}^k(\xi,t)) = 0, \quad (\xi,t) \in \widetilde{G}_h^k, \quad k = 4, 7.$$

Here

$$\Lambda_{(5.9)}(z^k(x,t)) = \Lambda_{(4.3)}(z^k(x,t)) - f(x,t), \quad (x,t) \in G_h^k, \quad k=1,$$

$$\Lambda_{(5.9)}(z^k(x,t)) \equiv \Big\{ \varepsilon\delta_{\overline{x}\widehat{x}} - \left(z^k(x,t)\right)^+ \delta_{\overline{x}} - \left(z^k(x,t)\right)^- \delta_x -$$
$$- \delta_{\overline{t}} - c(x,t) \Big\} z^k(x,t) - f(x,t), \quad (x,t) \in G_h^k,$$
$$x = x^i, \quad k = 1, \dots, 10, \quad k \neq 1,4,7,$$

$$\Lambda_{(5.9)}(\widetilde{z}^k(\xi,t)) \equiv \Big\{ \varepsilon t^{-2} \delta_{\overline{\xi}\widehat{\xi}} - t^{-1} \Big[\left(\widetilde{z}^k(\xi,t) - \xi\right)^+ \delta_{\overline{\xi}} + \left(\widetilde{z}^k(\xi,t) - \xi\right)^- \delta_\xi \Big] -$$
$$- \delta_{\overline{t}} - \widetilde{c}(\xi,t) \Big\} \widetilde{z}^k(\xi,t) - \widetilde{f}(\xi,t), \quad (\xi,t) \in \widetilde{G}_h^k,$$
$$\xi = \xi^i, \quad k = 4,7,$$

$(v(x,t))^+ = v^+(x,t)$, $(v(x,t))^- = v^-(x,t)$, $\delta_{\overline{x}\widehat{x}}\, z^k(x,t)$ and $\delta_{\overline{\xi}\widehat{\xi}}\, \widetilde{z}^k(\xi,t)$ are the second difference derivatives on nonuniform grids.

We write out conditions imposed on the functions $z^k(x,t)$ and $\widetilde{z}^k(\xi,t)$ on the sets S_h^k and $\widetilde{S}_h^k$, respectively. Set $\widehat{z}^k(x,t) = \widetilde{z}^k(\xi(x,t),t)$, $(x,t) \in \overline{\widehat{G}}{}_h^k$, $k=4,7$, where $\widehat{G}^0 = \{(x,t)\colon x = x(\xi,t),\ (\xi,t) \in \widetilde{G}^0\}$, $\widetilde{G}^0$ is one of the sets $\widetilde{S}_h^k$ or $\widetilde{G}_h^k$; $\overline{\widehat{G}}{}_h^k = \widehat{G}_h^k \cup \widehat{S}_h^k$, $k=4,7$. By $\overline{z}^k(x,t)$, $(x,t) \in \overline{G}^k$, $t \in \overline{\omega}_0^k$ for $k=2,\dots,10$, we denote the functions constructed by linear interpolation with respect to x from the values of the grid functions $z^k(x,t)$, $(x,t) \in \overline{G}_h^k$ and $\widehat{z}^k(x,t)$, $(x,t) \in \overline{\widehat{G}}{}_h^k$. The function $\overline{z}^1(x,t)$, $(x,t) \in \overline{G}^1$, $t \in \overline{\omega}_0^1$ is constructed by

$$\overline{z}^1(x,t) = \Big\{ \Big[z^1(x^{i-1},t) - W_{0(2.5)}(x^{i-1},t) \Big] (x^i - x) +$$
$$+ \Big[z^1(x^i,t) - W_{0(2.5)}(x^i,t) \Big] (x - x^{i-1}) \Big\} (x^i - x^{i-1})^{-1} + W_{0(2.5)}(x,t),$$
$$x \in [x^{i-1}, x^i], \quad (x^{i-1},t),\ (x^i,t) \in \overline{G}_h^1, \qquad t \in \omega_0^1, \quad t > 0; \tag{5.9b}$$
$$\overline{z}^1(x,0) = \varphi_0(x), \qquad x \in \overline{D},$$

where $x^{i+1} = x^i + h^1$, h^1 is the step-size of the grid $\overline{\omega}^1$.

Let us introduce the function $\overline{\overline{z}}(x,t)$, $(x,t) \in \overline{G}$, $t \in \overline{\omega}_0^0$, where

$$\overline{\omega}_0^0 = \left\{ \overline{\omega}_0^1 \bigcup \omega_0^4 \bigcup \omega_0^7 \bigcup \omega_0^{10} \right\}. \tag{5.9c}$$

Assume

$$\overline{\overline{z}}(x,t) = \overline{z}^k(x,t), \qquad (x,t) \in \overline{G}^k \setminus \Big\{ \bigcup_s \overline{G}^s \Big\},$$
$$\overline{\overline{z}}(x,t) = 2^{-1}(\overline{z}^k(x,t) + \overline{z}^s(x,t)), \qquad (x,t) \in \overline{G}^k \bigcap \Big\{ \bigcup_s \overline{G}^s \Big\},$$
$$t \in \omega_0^k, \quad s,\, k = 1,\dots,10, \quad s \neq k; \tag{5.9d}$$
$$\overline{\overline{z}}(x,0) = \varphi_0(x), \qquad x \in \overline{D}.$$

We define the functions $z^k(x,t)$ and $\tilde{z}^k(\xi(x,t),t) = \hat{z}^k(x,t)$ on the sets S_h^k and $\widehat{S}_h^k$ by the relations

$$\begin{aligned}
z^k(x,t) &= \varphi(x,t), && (x,t) \in S_h^k \bigcap S; \\
z^k(x,t) &= \overline{\overline{z}}(x,t), && (x,t) \in S_h^k \setminus S, \quad t \in \theta^k; \\
z^k(x,t) &= \overline{z}^s(x,t), && (x,t) \in \left\{S_h^k \setminus S\right\} \bigcap \left\{\bigcup_s \overline{G}^s\right\}, \quad t \in \omega_0^k, \quad k \neq 4,7; \\
\hat{z}^k(x,t) &= \varphi(x,t), && (x,t) \in \widehat{S}_h^k \bigcap S; \\
\hat{z}^k(x,t) &= \overline{\overline{z}}(x,t), && (x,t) \in \widehat{S}_h^k \setminus S, \quad t \in \theta^k; \\
\hat{z}^k(x,t) &= \overline{z}^s(x,t), && (x,t) \in \left\{\widehat{S}_h^k \setminus S\right\} \bigcap \left\{\bigcup_s \overline{G}^s\right\}, \quad t \in \omega_0^k, \quad k = 4,7,
\end{aligned} \tag{5.9e}$$

where $s,\ k = 1,\ldots,10,\ \ s \neq k,\ \ \theta^k = \overline{\omega}_0^k \setminus \omega_0^k$.

The finite difference scheme (5.9), (5.8) is the difference scheme based on domain decomposition that approximates problem (5.5), (5.1).

Consider that the steps of the grid $\overline{G}_h^1$ satisfy the condition $h^1(h_t^1)^{-3/2} \to 0$ for $N,\ N_0 \to \infty$, where h^1 and h_t^1 are, respectively, the steps of the grids $\overline{\omega}^1$ and $\overline{\omega}_0^1$ that generate the grid $\overline{G}_h^1$; $h^1 = 4l_{1(5.7)}N^{-1}$, $h_t^1 = t_{1(5.7)}N_0^{-1}$. Thus, under the condition

$$N_0^{-13/10} \leq \mathcal{O}(N^{-1}), \tag{5.10}$$

we have the following estimate for the finite difference scheme (5.9), (5.8):

$$|u(x,t) - \overline{\overline{z}}(x,t)| \leq M\left(\varepsilon^{1/6} + [N^{-1} + N_0^{-1}]^{5/63}\right), \quad (x,t) \in \overline{G}, \quad t \in \overline{\omega}_0^0, \tag{5.11}$$

that is, scheme (5.9), (5.8), (5.10) converges as $\varepsilon \to 0$ and $N,\ N_0 \to \infty$.

The estimate (5.11) is derived by using the majorant function technique (see, e.g., [14], [18]). When the values of ε are small, the derivatives of the solution to the boundary value problem are sufficiently large on the sets $\overline{G}^1$, $\overline{G}^4$ and $\overline{G}^7$. The convergence of the difference scheme on $\overline{G}^1$ is provided by using the fitted difference operator. On the set $\overline{G}^4$ with sufficiently small amount, the convergence is achieved by means of an especially small step of the grid with respect to the variables ξ and t. To approximate the boundary value problem on $\overline{\widetilde{G}}^7$, we use the special grid condensing in a neighborhood of the intervals $\xi = \varphi_0(-0)$ and $\xi = \varphi_0(+0)$. On the other sets $\overline{G}^2$, $\overline{G}^3$, $\overline{G}^5$, $\overline{G}^6$ and $\overline{G}^{10}$, the solution of the boundary value problem is relatively smooth; the convergence of the difference scheme on these sets is provided by natural decrease of the mesh widths in space and time as $N,\ N_0 \to \infty$. The existence of the solution to nonlinear problem (5.9), (5.8) follows from convergence of the solutions of the iterative grid process based on the alternating Schwartz method (see, e.g., [19]), which approximates the finite difference scheme (5.9), (5.8).

Theorem 4 *Assume that conditions* (A)–(D) *are valid, and let the solution of the boundary value problem* (1.2) *satisfy the estimates from Theorem* 2 *and the remark following it. Then, under condition* (5.10) *and for* $\varepsilon \to 0$, N, $N_0 \to \infty$, *the solution of the finite difference scheme* (5.9), (5.8) *converges to the solution of problem* (1.2). *The discrete solution satisfies* (5.11).

3. To find the approximate solution for boundary value problem (1.2), which is convergent uniformly with respect to the parameter, we use the following algorithm (we call it algorithm A). If $\varepsilon \geq [N^{-1}+N_0^{-1}]^{6/29}$, then the boundary value problem is approximated by the finite difference scheme (4.6), (4.1). But if $\varepsilon \leq [N^{-1}+N_0^{-1}]^{6/29}$, for the solution of the boundary value problem we use the finite difference scheme (5.9), (5.8). On the interval $[0,T]$, we introduce the mesh

$$\overline{\omega}_0^0 = \overline{\omega}^0_{0(5.12)}, \tag{5.12}$$

assuming $\overline{\omega}^0_{0(5.12)} = \overline{\omega}^0_{0(5.9c)}$ for scheme (5.9), (5.8), and $\overline{\omega}^0_{0(5.12)} = \overline{\omega}_{0(4.1a)}$ for scheme (4.6), (4.1). On the set $\overline{G}$, $t \in \overline{\omega}^0_{0(5.12)}$, we define the function

$$\overline{\overline{z}}(x,t) = \overline{\overline{z}}_{(5.13)}(x,t), \qquad (x,t) \in \overline{G}, \quad t \in \overline{\omega}_0^0, \tag{5.13}$$

where $\overline{\overline{z}}_{(5.13)}(x,t) = \overline{\overline{z}}_{(5.9d)}$ for scheme (5.9), (5.8). In the case of scheme (4.6), (4.1) for $(x,t) \in \overline{G}_{2(4.1b)}$, $t \neq t_1$, the function $\overline{\overline{z}}(x,t)$ is the linear interpolation in x from the values of the grid function $z(x,t)$, which is the solution of problem (4.6), (4.1). On the set $\overline{G}_{1(4.1b)}$, the function $\overline{\overline{z}}(x,t)$ is constructed by (5.9b), where $\overline{z}^1(x,t)$ and $z^1(x,t)$ is $\overline{\overline{z}}(x,t)$ and $z(x,t)$ respectively.

For the approximate solution we have the estimate

$$|u(x,t) - \overline{\overline{z}}(x,t)| \leq M \left[N^{-1} + N_0^{-1} \right]^{1/29}, \quad (x,t) \in \overline{G}, \quad t \in \overline{\omega}_0^0, \tag{5.14}$$

where $\overline{\overline{z}}(x,t) = \overline{\overline{z}}_{(5.13)}(x,t)$, $\overline{\omega}_0^0 = \overline{\omega}^0_{0(5.12)}$.

Theorem 5 *Let the assumptions of Theorems* 3 *and* 4 *be fulfilled. Then, under conditions* (4.7), (5.10), *the approximate solution of boundary value problem* (1.2), *which is constructed by the algorithm* A, *converges* ε*–uniformly for* N, $N_0 \to \infty$. *The discrete solution satisfies estimate* (5.14).

6 Grid approximations of the boundaries of a rarefaction wave

1. In this section we investigate the behavior of the boundaries of a rarefaction wave.

As $\varepsilon^{-1}t \to \infty$, the solution of problem (3.5) converges to the solution of problem (3.8) (see estimate (3.7)); the function $V_0(x,t)$ is the main term in the asymptotic expansion of the solution of problem (3.5) for $\varepsilon = o(t)$. The function $V_0(x,t)$ for $(x,t) \in \overline{G} \setminus S^\star$ consists of three smooth parts, divided by the rays $x = \varphi_0(-0)t$ and $x = \varphi_0(+0)t$, which are the boundaries of a nonviscous rarefaction wave. We denote these boundaries by $x = s_0^-(t)$, $x = s_0^+(t)$ (the left and right boundaries of wave). In the case of problem (3.5) for $\varepsilon = o(t)$, it is possible to pick out three parts in the interval $\overline{D}$, on each of them the first derivative with respect to x (for fixed t) is almost constant except for the narrow transient layers which divide these parts of the interval. The width of the transient layers goes to zero as $\varepsilon \to 0$.

Note that $|V(x,t) - V_0(x,t)| \geq m$, $x = s_0^\pm(t)$, $t = \varepsilon$ for any value of $\varepsilon \in (0,1]$. This means that the function $V_0(x,t)$ does not approximate the solution of problem (3.5) in a neighborhood of the point $S^\star$ (for $t \leq \mathcal{O}(\varepsilon)$), when the parameter goes to zero.

We call the curves $x = s^-(t)$ and $x = s^+(t)$ the boundaries of the "viscous" rarefaction wave in the case of problem (3.5), where the functions $s^-(t)$ and $s^+(t)$ are defined by the relations

$$\frac{\partial}{\partial x}V(x^1,t),\ \frac{\partial}{\partial x}V(x^2,t) \ < \ \frac{\partial}{\partial x}V(s^-(t),t) = \frac{\partial}{\partial x}V(s^+(t),t) \equiv 2^{-1}\max_{\overline{D}}\frac{\partial}{\partial x}V(x,t),$$
$$x^1 < s^-(t), \quad x^2 > s^+(t), \quad t \in (0,T]; \quad s^-(t) = s^+(t) = 0, \quad t = 0.$$

On the left and right boundaries of the "viscous" wave for each $t > 0$, the first derivative with respect to x is equal to the one-half of its maximum value. Taking account of (3.7), we establish the following estimates for the functions $s^-(t)$ and $s^+(t)$:

$$|s^-(t) - s_0^-(t)|, \quad |s^+(t) - s_0^+(t)| \ \leq \ M\,\varepsilon^{1/2}\,t^{1/3}, \quad t \in [0,T],$$

that is, the functions $s_0^-(t)$, $s_0^+(t)$ are the main terms in the asymptotic expansions of the functions $s^-(t)$, $s^+(t)$. Such behavior of $s^-(t)$ and $s^+(t)$ is not contrary to the fact that the function $V_0(x,t)$ does not approximate the function $V(x,t)$ in a neighborhood of the point $S^\star$ as $\varepsilon \to 0$; both the functions $s^-(t)$, $s^+(t)$ and the functions $s_0^-(t)$, $s_0^+(t)$ go to zero as $t \to 0$.

It follows from estimate (3.14) that the function $u_{0(3.13)}(x,t)$ is the main term in the asymptotic expansion of the solution of problem (1.2). From estimate (3.7) it follows that the function $V(x,t)$ is approximated well by the function $V_0(x,t)$ for sufficiently small magnitudes of εt^{-1}. For small values of ε and t the function $V(x,t)$ is close to the function $W_0(x,t)$. The derivative $(\partial/\partial x)u(x,t)$ satisfies the estimate

$$\max_{\overline{D}}\frac{\partial}{\partial x}\,u(x,t) \ \geq \ m^1\,\rho^{-1}(t,\varepsilon), \quad t \in (0,\delta].$$

Moreover,

$$\frac{\partial}{\partial x}u(x,t) \ \geq \ m^2\,\rho^{-1}(t,\varepsilon), \quad x \in \left[\varphi_0(0)t - m^3\rho(t,\varepsilon),\ \varphi_0(0)t + m^3\rho(t,\varepsilon)\right], \quad t \in (0,\delta]$$

where $\rho(t,\varepsilon) = t^{1/2}(\varepsilon^{1/2} + t^{1/2})$.

In the case of problem (1.2) we define the curves $x = s^-(t)$, $x = s^+(t)$, i.e., the left and right boundaries of the "viscous" wave for $t \in [0, \delta]$, where δ is a sufficiently small number, by the relations

$$\frac{\partial}{\partial x}u(x^1,t),\ \frac{\partial}{\partial x}u(x^2,t) \ < \ \frac{\partial}{\partial x}u(s^-(t),t) = \frac{\partial}{\partial x}u(s^+(t),t) \equiv 2^{-1} \max_{\overline{D}} \frac{\partial}{\partial x}u(x,t),$$

$$x^1 < s^-(t), \quad x^2 > s^+(t), \quad t \in (0,T]; \quad s^-(t) = s^+(t) = 0, \quad t = 0. \qquad (6.1)$$

Here $u(x,t) = u_{(1.2)}(x,t)$ is the solution of problem (1.2), $s^\pm(t) = s^\pm_{(1.2,\,6.1)}(t) \equiv s^\pm_{(6.1)}(t)$. Note that, for sufficiently small fixed t, the derivative $(\partial/\partial x)u(x,t)$ in the nearest neighborhood of the set $[s^-(t),\ s^+(t)]$ strictly decreases when a distance to this set increases. The value of δ is chosen such that the derivative $(\partial/\partial x)u(x,t)$ outside a sufficiently small neighborhood of the set $[s^-(t),\ s^+(t)]$ is essentially less than the value of $(\partial/\partial x)u(s^\pm(t),t)$ for $t \in (0,\delta]$.

In the case of problem (1.4), we denote the left and right boundaries of wave by $x = s_0^-(t)$, $x = s_0^+(t)$, where

$$s_0^\pm(t) = s^\pm_{0\,(1.4,\,6.2)}(t) \equiv s^\pm_{0\,(6.2)}(t). \qquad (6.2)$$

Such boundaries are the curves on $\overline{G}$ that separate the domains of smoothness of the generalized solution.

We have the following estimate

$$|s^-(t) - s_0^-(t)|, \quad |s^+(t) - s_0^+(t)| \ \le\ M\,\varepsilon^{1/2}, \quad t \in [0,\ \delta],$$

that is, the functions $s_0^-(t)$, $s_0^+(t)$ are the main terms in the asymptotic expansion of the functions $s^-(t) = s^-_{(6.1)}(t)$, $s^+(t) = s^+_{(6.1)}(t)$.

2. When constructing grid approximations of the functions $s^-(t)$, $s^+(t)$, $t \in [0,\delta]$, we use the approximate solution generated by the algorithm A. We define the functions $s^{h-}(t)$, $s^{h+}(t)$ by the relations

$$\delta_x^*\,\overline{\overline{z}}(x^1,t),\ \delta_x^*\,\overline{\overline{z}}(x^2,t) \ < \ \delta_x^*\,\overline{\overline{z}}(s^{h-}(t),t) = \delta_x^*\,\overline{\overline{z}}(s^{h+}(t),t) \equiv 2^{-1} \max_{\overline{D}} \delta_x^*\,\overline{\overline{z}}(x,t),$$

$$x^1 < s^{h-}(t), \quad x^2 > s^{h+}(t), \quad t{>}0; \quad s^{h-}(t) = s^{h+}(t) = 0, \quad t{=}0; \quad t \in \overline{\omega}_0^0, \quad t \le \delta, \qquad (6.3)$$

where $\overline{\overline{z}}(x,t) = \overline{\overline{z}}_{(5.13)}(x,t)$, $\overline{\omega}_0^0 = \overline{\omega}^0_{0(5.12)}$, $\delta_x^*\,v(x,t) = 2^{-1}l^{-1}(v(x{+}l,t) - v(x{-}l,t))$, and also

$$l = l_{(6.4)}(t,\varepsilon,N,N_0) = M\,\rho(t,\varepsilon)\ [\,N + N_0\,]^{-1/29}\,. \qquad (6.4)$$

For the functions $s^{h-}(t) = s^{h-}_{(6.3,6.4)}(t)$, $s^{h+}(t) = s^{h+}_{(6.3,6.4)}(t)$, which are defined by (6.3), (6.4), we have the estimate

$$|s^-(t) - s^{h-}(t)|,\ |s^+(t) - s^{h+}(t)| \le M[N + N_0]^{-1/29}, \quad t \in \overline{\omega}_0^0, \quad t \le \delta. \qquad (6.5)$$

Hence, the functions $s^{h-}(t)$, $s^{h+}(t)$ converge to the functions $s^-(t) = s^-_{(6.1)}(t)$, $s^+(t) = s^+_{(6.1)}(t)$ uniformly with respect to the parameter. When we choose the value of l in (6.3) from the condition

$$l = l_{(6.6)}(t, \varepsilon, N, N_0) = M\left(\rho(t,\varepsilon)\big[N + N_0\big]^{-1/29} + \varepsilon^{1/2}\, t^{1/2}\right), \tag{6.6}$$

we obtain the following estimate

$$|s_0^-(t) - s^{h-}(t)|,\ |s_0^+(t) - s^{h+}(t)| \le M\left(\varepsilon^{1/2} + \big[N + N_0\big]^{-1/29}\right), \quad t \in \overline{\omega}_0^0,\ \ t \le \delta, \tag{6.7}$$

that is, for $\varepsilon \to 0$ and N, $N_0 \to \infty$ the functions $s^{h-}(t) = s^{h-}_{(6.3,6.6)}(t)$ and $s^{h+}(t) = s^{h+}_{(6.3,6.6)}(t)$, which are defined by (6.3), (6.6), converge to the functions $s_0^-(t) = s^-_{0(6.2)}(t)$, $s_0^+(t) = s^+_{0(6.2)}(t)$ that correspond to the limit (for $\varepsilon = 0$) problem (1.4).

Theorem 6 *Let the assumptions of Theorem* 5 *be fulfilled. Then the functions* $s^{h-}_{(6.3,6.4)}(t)$, $s^{h+}_{(6.3,6.4)}(t)$, $t \in \overline{\omega}^0_{0(5.12)}$, $t \le \delta$ *converge, as* N, $N_0 \to \infty$, *to the functions* $s^-_{(6.1)}(t)$, $s^+_{(6.1)}(t)$ *uniformly with respect to the parameter* ε*; the functions* $s^{h-}_{(6.3,6.6)}(t)$, $s^{h+}_{(6.3,6.6)}(t)$, $t \in \overline{\omega}^0_{0(5.12)}$, $t \le \delta$ *converge, as* N, $N_0 \to \infty$ *and* $\varepsilon \to 0$, *to the functions* $s^-_{0(6.2)}(t)$, $s^+_{0(6.2)}(t)$. *The estimates* (6.5) *and* (6.7) *hold, respectively, for the functions* $s^{h-}_{(6.3,6.4)}(t)$, $s^{h+}_{(6.3,6.4)}(t)$ *and* $s^{h-}_{(6.3,6.6)}(t)$, $s^{h+}_{(6.3,6.6)}(t)$.

7 Remarks and generalizations

1. Sufficiently intricate construction of the difference scheme is due to the behavior of the singular part of the solution of the boundary value problem. For sufficiently small values of the parameter, the solution of the problem for x, $t = \mathcal{O}(\varepsilon)$ is close to the solution of the linear heat equation $(\varepsilon\partial^2/\partial x^2 - \partial/\partial t)u(x,t) = 0$ with a discontinuous initial condition $u(x,0) = \eta(x)$, where $\eta(x)$ is a step function, $\eta(x) = \varphi_0(-0)$ for $x < 0$ and $\eta(x) = \varphi_0(+0)$ for $x > 0$. Natural variables for x, $t = o(\varepsilon)$ are the scaling (self-similar) variable $\zeta = \varepsilon^{-1/2}xt^{-1/2}$ and $\tau = t$. When x, $t = \mathcal{O}(\varepsilon^\nu)$, where $\nu > 0$ is a sufficiently small number, the solution of the problem for $t \ge m\varepsilon$ is close to the generalized solution of the quasilinear hyperbolic equation $(-u(x,t)\,\partial/\partial x - \partial/\partial t)u(x,t) = 0$ with a discontinuous initial condition. Natural variables for $\varepsilon = o(t)$, x, $t = o(\varepsilon^\nu)$ are the scaling variable $\xi = xt^{-1}$ and $\tau = t$. For finite and not too small values of the parameter, the singular part of the solution is close to the solution of a singular perturbed heat equation with a discontinuous initial condition if the value of t is sufficiently small, $t = o(1)$.

2. If boundary layers are present in the solution, we use piecewise uniform grids condensing by a special way in a neighborhood of the boundary layers (see, e.g., [16]).

3. We discuss the motivation of conditions (A)–(D) and (3.3), (3.12c), and the principles for a choice of the data to problem (1.2) that ensure the validity of these conditions.

In the case of a discontinuous initial condition only the positive jump of the initial function (see condition (1.3)) initiates a rarefaction wave. For $c(x,t) = f(x,t) \equiv 0$, $(x,t) \in \overline{G}$, if we choose a smooth function $\varphi_0(x)$ strictly increasing on the parts of its continuity, it is not difficult to satisfy conditions (A), (3.3), (3.12c) for $(x,t) \in \overline{G}$ (it is suitable to extend the function $\varphi_0(x)$ onto the x–axis and to consider equations (1.2a) and (1.4a) on the strip H). The strict increasing, in x, of the solution of the reduced equation is kept for sufficiently small values of time, if the functions $c(x,t)$ and $f(x,t)$ are bounded on $\overline{G}$. In this case the characteristics of equation (1.4a) satisfy condition (B). The existence of a small viscosity is little manifested in a neighborhood of the smooth parts of the solution to the reduced problem. The viscous solution keeps its increasing with respect to x for small values of time, that is, the solution of problem (1.2) satisfies condition (C).

In that case when the solution of the discrete problem is nondecreasing with respect to x (i.e., condition (D) holds), it converges ε–uniformly to the solution of the boundary value problem, which strictly increases with respect to x (for suitable data of problem (1.2)). The constructed difference schemes are monotone (for both schemes (4.6), (4.1) and (5.9), (5.8), just as for the boundary value problem in the form (1.2) and (5.5), (5.1) under consideration, the appropriate variants of the maximum principle are valid). This property of the schemes also ensures the applicability of the majorant function technique to the analysis of their convergence. Therefore, condition (D) seems sufficiently natural. The validity of this fact is easily controlled in the computing process. If conditions (A), (C), (3.3b), (3.12c) are violated, the interior (transient) layers may appear, where the solution varies by a finite quantity. In that case when such transient layers appear, the statements of Theorem 3, and also Theorems 4 and 5 hold outside a sufficiently small m-neighborhood (in the variables x, t) of these layers.

4. Note that all the above constructions of the special difference schemes are preserved in the case of

$$L(u(x,t)) \equiv \left\{ \varepsilon \frac{\partial^2}{\partial x^2} - u(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right\} u(x,t) = g(x,t,u(x,t)), \quad (x,t) \in G,$$

if the following condition holds:

$$\left| \frac{\partial}{\partial u} g(x,t,u) \right| \le M_1, \quad (x,t) \in \overline{G}, \quad |u| \le M_2.$$

5. The finite difference schemes (4.6), (4.1) and (5.9), (5.8) are nonlinear. To find the solution of these schemes, it is possible to use iterative algorithms similar to those developed in [17], [20].

6. To solve the discrete problem (5.9), (5.8), one can use the alternating Schwartz method described in [18].

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