The Phase Diagrams of the Schwinger and Gross-Neveu Models with Wilson Fermions

R. Kenna[a]* C. Pinto[a,b] and J.C. Sexton[a,b]

[a]School of Mathematics, Trinity College Dublin, Ireland,

[b] Hitachi Dublin Laboratory, Dublin, Ireland

A new method to analytically determine the partition function zeroes of weakly coupled theories on finite–size lattices is developed. Applied to the lattice Schwinger model, this reveals the possible absence of a phase transition at fixed weak coupling. We show how finite–size scaling techniques on small or moderate lattice sizes may mimic the presence of a spurious phase transition. Application of our method to the Gross–Neveu model yields a phase diagram consistent with that coming from a saddle point analysis.

1. Introduction

There has recently been considerable discussion on the phase structure of theories with Wilson fermions. A system of free Wilson fermions exhibits a second order phase transition at $\kappa = 1/2d$, κ being the hopping parameter and d the dimensionality. Discussions concern the extent to which this phase transition persists when a bosonic (e.g. gauge) field is included. In the Schwinger model, with inverse gauge coupling squared β , the expectation is that there is a line of phase transitions in the (β, κ) plane extending to the strong coupling limit $\beta = 0$ [1]. This critical line is also expected to recover massless physics and therein lies the importance in determining its nature and position [2,3]

The phase diagram of the Schwinger model has been numerically determined by two groups, which, while in rough agreement regarding the location of the critical line, differ in their conclusions regarding its quantitive critical properties. Using Lee–Yang zeroes, finite–size scaling and other techniques the results of [2] support the free boson scenario, where the model lies in the same universality class as the Ising model, with critical exponents $\nu = 1$, $\alpha = 0$. This is not in agreement with [3] where finite–size scal-

ing of Lee–Yang zeroes on larger lattices provides evidence for $\nu = 2/3$, $\alpha = 2/3$.

We analytically determine the phase structure in the weakly coupled regime. We find that the zeroes in the Schwinger model with fixed weak gauge coupling display unusual behaviour, do not accumulate on the real axis and may not, in fact, lead to a phase transition. The behaviour of these zeroes is, however, such as to mimic the appearance of a phase transition when a finite–size analysis is restricted to small or moderate lattices.

Applying our method to the $d = 2$ Gross–Neveu model yields results consistent with Aoki's saddle point analysis [4] and this consistency adds confidence to our new analytic approach.

2. Lattice Schwinger Model

We consider a $d = 2$ lattice of spacing a and N sites in each direction. For the fermion fields, we impose antiperiodic (periodic) boundary conditions in the 1- (2-) direction. The Fourier transformed fermion fields then have momenta $p_\mu = (2\pi/Na)\hat{p}_\mu$ where $\hat{p}_1$ is half–integer and $\hat{p}_2$ is integer. The action is $S_{\mathrm{QED}} = S_G + S_F^{(W)}$, where $S_F^{(W)} = \sum_{m,n} \bar{\psi}_m M_{m,n}^{(W)} \psi_n$ and

$$M_{m,n}^{(W)} = \frac{\delta_{m,n}}{2\kappa} - \frac{1}{2} \sum_\mu \left\{ (1 - \gamma_\mu)U_\mu(m)\delta_{m+\hat{\mu},n} \right.$$
$$\left. +(1 + \gamma_\mu)U_\mu^\dagger(n)\delta_{m-\hat{\mu},n} \right\} \quad . \tag{1}$$

This fermion matrix consists of free and interacting parts, $M^{(W)} = M^{(0)} + M^{(\text{int})}$ with $M^{(0)}$ given by $U_\mu = 1$ in (1). Here $U_\mu(n) = \exp(ie_0 a A_\mu(n))$ is the link variable. The partition function is $Z = \int \mathcal{D}U \exp(\{-S_{\text{G}} - S_{\text{F}}^{(W)}\})\langle \det M^{(W)}\rangle$, the integration over fermions having been performed. The gauge action is $S_G = \beta \sum_P \left[1 - \frac{1}{2}(U_P + U_P^\dagger)\right]$, where U_P is the product of link variables around a plaquette, $\beta = 1/e_0^2 a^2$ and $\langle \mathcal{O}\rangle$ is a pure gauge expectation.

For $\beta = \infty$, the partition function is simply proportional to $\det M^{(0)}$. Then the free partition function can be written in terms of its eigenvalues $\lambda_\alpha^{(0)}(\hat{p}) = 1/2\kappa - \sum_{\mu=1}^2 \cos p_\mu a + i(-)^\alpha \sqrt{\sum_{\mu=1}^2 \sin^2 p_\mu a}$ which are either 2–fold ($\hat{p}_2 = 0$ or $-N/2$) or 4–fold degenerate. In this case, and with $\eta = 1/2\kappa$, the zeroes are $\eta_\alpha^{(0)}(\hat{p}) = \sum_{\mu=1}^2 r \cos p_\mu a - i(-)^\alpha \sqrt{\sum_{\mu=1}^2 \sin p_\mu a}$. The lowest zero with finite real part in κ has $\hat{p} = (\pm 1/2, 0)$. Pinching of the positive real finite hopping parameter axis occurs at $\kappa_c = 1/2d$ and application of finite–size scaling to the imaginary parts of the zero gives the critical exponent $\nu = 1$ ($\alpha = 0$ then follows from hyperscaling). Therefore the free fermion model is in the same universality class as the Ising model in two dimensions and describes free bosons.

3. The Weak Coupling Expansion

The weak coupling expansion is obtained by expanding the link variables $U_\mu(n)$ as a power series in e_0. The fermion determinant can then be expanded giving the 'traditional' form for the weak coupling expansion (see e.g., [5]). This expansion is analytic in η with poles at $\eta = \eta_i^{(0)}$ [6].

An alternative formulation of the partition function may be obtained by writing the fermion matrix as $M^{(W)} = \eta + H$. The determinant $\det M^{(W)} = \det(\eta + H)$ is a polynomial in η for finite lattice size. Therefore its pure gauge expectation value is also a polynomial in η and may be written $\langle \det M^{(W)}\rangle = \prod_{i=1}^{dN^d}(\eta - \eta_i)$. Here η_i represent $\eta_\alpha(\hat{p})$ and are the zeroes in the complex $1/2\kappa$ plane. We may write a 'multiplicative' weak

coupling expansion as

$$\frac{\langle \det M^{(W)}\rangle}{\det M^{(0)}} = \prod_{i=1}^{dN^d}\left(1 - \frac{\Delta_i}{\eta - \eta_i^{(0)}}\right) \quad , \qquad (2)$$

where $\Delta_i = \eta_i - \eta_i^{(0)}$ are the shifts that occur in the zeroes when the gauge field is turned on. Note that (2) is analytic in η with poles at $\eta_i^{(0)}$. Since the multiplicative expression (2) must be the same as the 'traditional' weak coupling expansion, the residues of the poles must be equal.

Equating these residues gives the the shifts in the positions of the zeroes in the two fold degenerate case (the case of four fold degeneracies is more complicated but we expect the lowest zeroes to be erstwhile two fold degenerate). In this way, the shifts in the positions of the lowest zeroes are determined to $\mathcal{O}(e^2)$. Our calculation of pure gauge expectation values is done in Feynman gauge.
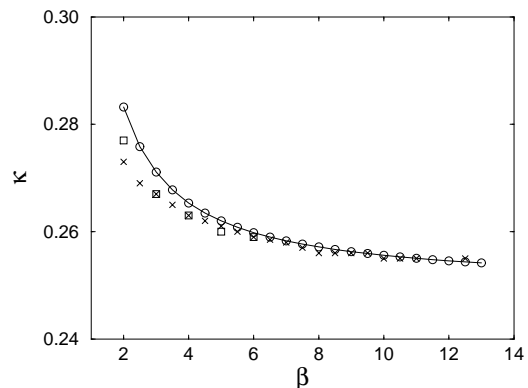
4. Results and Conclusions

Figure 1. The would-be phase diagram coming from the real parts of the lowest zeroes at $N = 24$.

A standard numerical technique for determination of the phase diagram is to approximate the critical point by the real part of the lowest zero for some large lattice. Plotted against β this approximates the phase diagram. In Figure 1 we present such a plot for $N = 24$ (circles) to compare with the results of [3] (crosses) also at $N = 24$. The phase diagram of [2] for the Schwinger model with *two* fermion flavours coming from a separate PCAC based analysis is also included (squares) for comparison.

Figure 2 is a finite–size scaling plot for the lowest zero at $\beta = 10$ and $N = 8 - 62$ coming from our weak coupling analysis (circles). The corresponding data from the numerical analysis of [3] are also included (crosses). The two lines are linear fits to the first and second zeroes for lattice sizes $16, 20, 24$ which are those analysed in [3]. These yield slopes -1.5 and -1.4 respectively, corresponding to $\nu \approx 2/3$. A similar plot may be made to compare with the numerical results of [2] who use $\beta = 5$ and $N = 2, 4, 8$. There, the best linear fit gives ν compatable with 1. Fitting to our small lattice data also gives $\nu \approx 1$. Thus
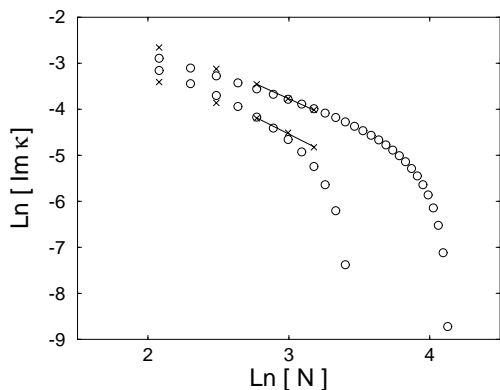

Figure 2. Finite size scaling of the imaginary parts of the first two Lee–Yang zeroes at $\beta = 10$.

we find that confining the finite–size scaling analysis to the range $N = 2 - 8$ yields $\nu = 1$ in agreement with [2] while a corresponding analysis with $N = 8$–24 gives $\nu = 2/3$ in agreement with [3]. It is clear from the figure, however, that the curve does not in fact settle to a finite–size scaling line. Instead as N increases, the lowest zeroes cross the real axis. The first two zeroes therefore fail to accumulate and do not contribute to critical behaviour [7]. These zeroes are isolated singularities, having measure zero in the thermodynamic limit. Although the possibility of existence of isolated singularities and non–accumulation of partition function zeroes has been known for a long time [7], this is to our knowledge the first instance where such behaviour has been observed.

We have also applied our new analytic technique to the two dimensional Gross–Neveu model

[8]. There we find the expected accumulation of zeroes and a weakly coupled phase diagram which is consistent with that determined by Aoki using a saddle point approach [4].

In conclusion, we have developed a new method to analytically determine the Lee–Yang zeroes of weakly coupled theories. In the free case, there is a phase transition precipitated by the accumulation of Lee–Yang zeroes on the real hopping parameter axis. In the weakly coupled lattice Schwinger model at fixed β, this accumulation no longer occurs for the first couple of zeroes. Instead, the movement of these zeroes for small and moderately sized lattices mimics phase transition like behaviour. As the lattice size becomes large, however, these zeroes move across the real axis, and do not give rise to a phase transition. In the Gross–Neveu case, our phase diagram is consistent with Aoki's saddle point analysis [8].

REFERENCES

1. M. Salmhofer, Nucl. Phys. B 362 (1991) 641; H. Gausterer, C.B. Lang and M. Salmhofer, Nucl. Phys. B 388 (1992) 275; H. Gausterer and C.B. Lang, Nucl. Phys. B 455 1995 785; F. Karsch, E. Meggiolaro and L. Turko Phys. Rev. D 51 (1995) 6417; C. Gattringer, hep-lat/9903021 (to appear in Nucl. Phys. B).
2. H. Gausterer and C.B. Lang, Phys. Lett. B 341 (1994) 46; Nucl. Phys. B (Proc. Supl.) 34 (1994) 201; I. Hip, C.B. Lang and R. Teppner, Nucl. Phys. B (Proc. Supl.) 63 (1998) 682.
3. V. Azcoiti, G. Di Carlo, A. Galante, A.F. Grillo and V. Laliena, Phys. Rev. D 50 (1994) 6994; ibid. 53 (1996) 5069.
4. S. Aoki, Phys. Rev. D 30 (1984) 2653.
5. H.J. Rothe, *Lattice Gauge Theories* (World Scientific, Singapore, 1997).
6. R. Kenna, J.C. Sexton and C. Pinto, hep-lat/9812004.
7. R. Abe, Prog. Theor. Phys. 38 (1967) 322; M. Salmhofer, Nucl. Phys. B, Proc. Suppl. 30 (1993) 81; Helv. Phys. Acta 67 (1994) 257.
8. R. Kenna, J.C. Sexton and C. Pinto, in preparation.